C(0,0)
C(1,0) C(1,1)
C(2,0) C(2,1) C(2,2)
C(3,0) C(3,1) C(3,2) C(3,3)

1 3 3 1

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

The Binomial Theorem:

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

C(0,0)

C(1,0) C(1,1)

C(2,0) C(2,1) C(2,2)

C(3,0) C(3,1) C(3,2) C(3,3)

C(4,0) C(4,1) C(4,2) C(4,3) C(4,4)

1 4 6 4 1

The Binomial Theorem:

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$\sum_{k=0}^{n} C(n, k) = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

The Binomial Theorem:

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

n = 4

```
0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
0 1 0 1
0 1 1 0
0 1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1
```

Permutations and Combinations with Repetition

How many strings of length r can we form from the uppercase letters of the English alphabet, if repetition is allowed?

26^r

Set of size n, selecting r items, $0 \le r \le n$

	Permutations (ordered)	Combinations (unordered)
Without repetition	$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$ $= \frac{n!}{(n-r)!}$	$C(n, r) = \frac{P(n, r)}{r!}$ $= \frac{n!}{r!\,(n-r)!}$
With repetition	n^r	

Combinations with Repetition

Consider 7 kinds of bills: \$1, \$2, \$5, \$10, \$20, \$50, \$100

Problem: Suppose I have a bag with lots of bills in it, and I pull out 5 bills. How many different combinations could there be?

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Equivalent problem: Suppose I have 7 empty bins, labeled with the denominations. I'm going to place 5 blank bills in them, to be printed later. How many ways can I distribute the 5 bills?

Equivalent problem: How many sets can I form by selecting 5 items, with repetition allowed, from a set of 7 items?

Equivalent problem: How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 5 \quad \text{where } 0 \le x_i \le 5 \text{ ?}$$

Combinations with Repetition

Equivalent problem: Suppose I have 7 empty bins, labeled with the denominations. I'm going to place 5 blank bills in them, to be printed later. How many ways can I distribute the 5 bills?

*		* *		*		*
\$1	\$2	\$5	\$10	\$20	\$50	\$100

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* | | * * | | * | | *

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* | | * * | | * | | *

* | | * * | | * | | *

\- - - - - - - - - - -

How many ways can we put 5 stars in these 11 slots?

Combinations with Repetition

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* | | * * | | * | | *

* | | * * | | * | | *

\- - - - - - - - - - -

How many ways can we put 5 stars in these 11 slots?

There are "11 choose 5" ways, i.e, $C(11, 5)$.

So there are $C(n+r-1, r)$ r-combinations from a set of n elements when repetition is allowed.

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* | | ** | | * | | | *

* | | * * | | * | | *

\- \- \- \- \- \- \- \- \- \- \-

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There are "11 choose 5" ways, i.e, $C(11, 5)$.

So there are $C(n+r-1, r)$ r-combinations from a set of n elements when repetition is allowed. Since we could also think of placing the bars, we have

$$C(n+r-1, r) = C(n+r-1, n-1)$$

Set of size n, selecting r items, $0 \le r \le n$

	Permutations (ordered)	Combinations (unordered)
Without repetition	$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$	$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!\,(n-r)!}$
With repetition	n^r	$C(n+r-1, r)$ or $C(n+r-1, n-1)$

More Useful Formulas

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n+r}{r} = \binom{n+r}{n}$$

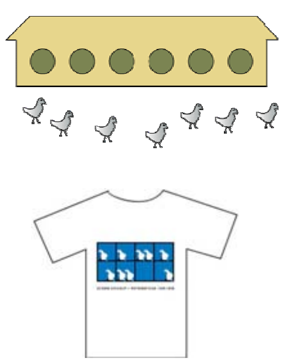
Without answering either question, why do these two questions have the same answer?

- How many sets of size 2 can be made choosing elements from {1, 2, 3, 4, 5, 6, 7, 8, 9}?
- How many sets of size 7 can be made choosing elements from {1, 2, 3, 4, 5, 6, 7, 8, 9}?

Because every solution to one of the questions also specifies a solution to the other.

{3, 8} ⇔ {1, 2, 4, 5, 6, 7, 9}
{5, 7} ⇔ {1, 2, 3, 4, 6, 8, 9}

$$\binom{n}{r} = \binom{n}{n-r}$$



<u>The Pigeonhole Principle</u>:

If k+1 or more objects are placed in k boxes, then there is at least one box containing two or more of the objects.

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If k+1 or more objects are placed in k boxes, then there is at least one box containing two or more of the objects.

Among any group of 367 people, there must be at least two with the same birthday since there are only 366 possible birthdays.

In any group of 27 English words, there must be at least two that start with the same letter, since there are 26 letters in the alphabet.

The Generalized Pigeonhole Principle:

If N objects are placed in k boxes, then there is at least one box containing at least ceil(N/k) objects.

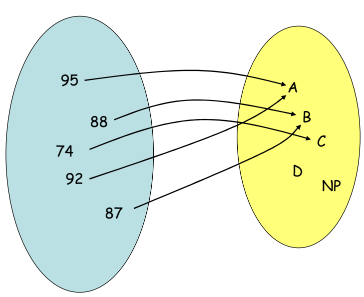
Among 100 people there are at least ceil(100/12) = 9 who were born in the same month.

What is the minimum number of students required in a class to be sure that at least six will receive the same grade, given the five possible grades A, B, C, D, NP?

Functions

Suppose that after curving, we round all final averages to whole numbers, and the grades A, B, C, D, and NP are assigned in the usual way: 100 – 90 is an A, 89 – 80 is a B, etc.

We would say that the grade assignment scheme gives us a mapping from number grades to letter grades, and that we have a function that maps numbers to grades.



Functions

Definitions

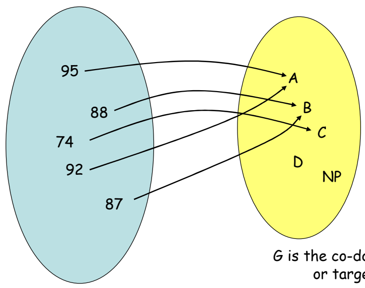
A function f is a mapping from a set D to a set T with the property that for each element d in D, the function f maps d to a single element of T, denoted f(d).

D is called the domain of f, and T is called the target or co-domain.

Thus, we write f: D $\rightarrow$ T.

We also say that f(d) is the image of d under f, and we call the set of all images the range R of f.

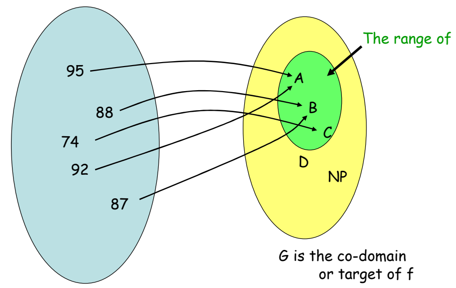
f : S→G



G is the co-domain
or target of f

S is the domain of f

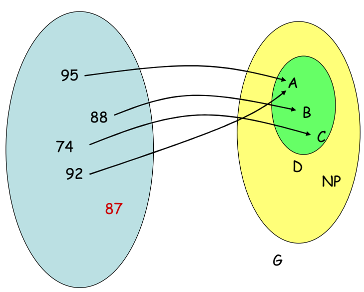
f : S→G



G is the co-domain
or target of f

S is the domain of f

This is NOT a function from S to G:

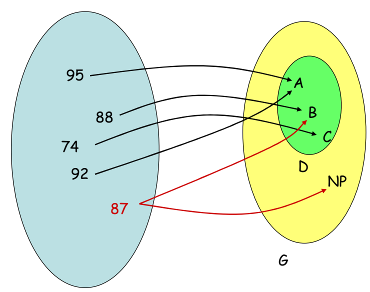


G

S

To be a function, every member of the domain must be mapped to a member of the co-domain.

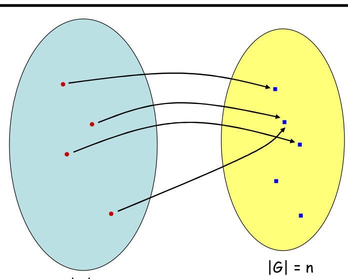
This is NOT a function from S to G:



G

S

To be a function, no member of the domain may map to more than one member of the co-domain.



|G| = n

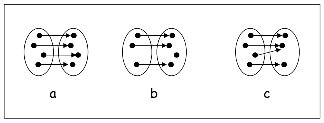
|S| = r

How many functions are there from S to G?

Set of size n, selecting r items, $0 \le r \le n$

	Permutations (ordered)	Combinations (unordered)
Without repetition	$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$ $= \frac{n!}{(n-r)!}$	$C(n, r) = \frac{P(n, r)}{r!}$ $= \frac{n!}{r!\,(n-r)!}$
With repetition	n^r	$C(n+r-1, r)$ or $C(n+r-1, n-1)$

Functions

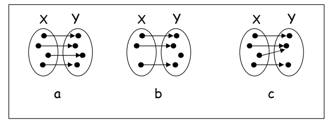


A function is said to be **onto** if its range is equal to its target (a and c). Onto functions are also called **surjections**.

A function is said to be **one-to-one** if it maps distinct elements of the domain to distinct elements of the range (a and b). One-to-one functions are also called **injections**.

A function is said to be a **one-to-one correspondence** if it is both one-to-one and onto (a). Such a function is also called a **bijection**.

Functions



A function is said to be **onto** if its range is equal to its target (a and c).

Surjective: $\forall y \in Y\ (\exists x \in X\ (f(x) = y))$

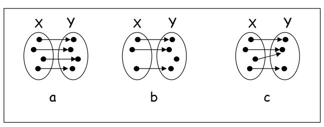
A function is said to be **one-to-one** if it maps distinct elements of the domain to distinct elements of the range (a and b).

Injective: $\forall x_1, x_2 \in X\ (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$

A function is said to be a **one-to-one correspondence** if it is both one-to-one and onto (a).

Bijective: Surjective and Injective

Functions

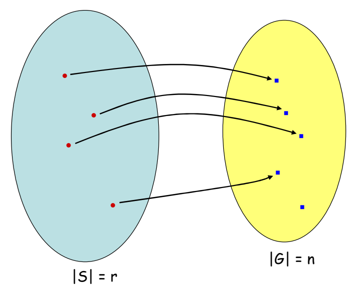


For a function $f{:}X \rightarrow Y$, given an element $y \in Y$, a pre-image of y (under f) is an element $x \in X$ such that $y = f(x)$.

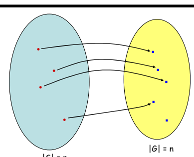
f is a surjection iff every element of y has at least one pre-image (a and c)

f is an injection iff every element of Y has at most one pre-image (a and b)

f is a bijection iff every element of y has precisely one pre-image (a)



How many injections are there from S to G?



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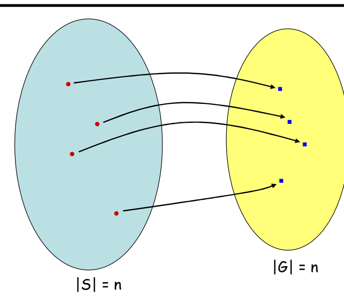

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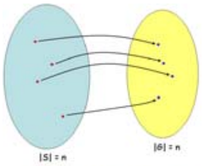

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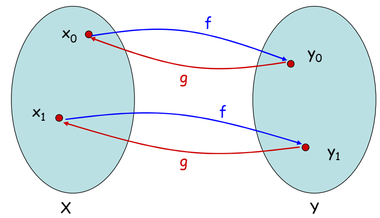
Inverse

A function $f: X \to Y$ is called invertible if and only if there exists a function $g : Y \to X$ such that

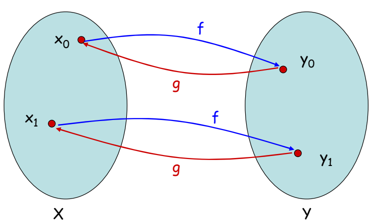
$$y = f(x) \leftrightarrow x = g(y) \text{ for all } x \in X \text{ and for all } y \in Y.$$

We call g the inverse of f and write $g = f^{-1}$.



Theorem: A function $f: X \to Y$ is invertible if and only if it is a bijection, and if f is invertible, then the inverse is unique.

bijection → invertible



Bijection

Surjective: Everything has an incoming arrow (onto)

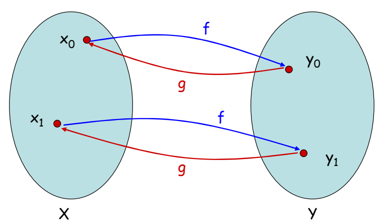
AND

Injective: Nothing has more than one incoming arrow (one-to-one)

We can construct an inverse: if f(x) = y, then g(y) = x.

Theorem: A function $f: X \to Y$ is invertible if and only if it is a bijection, and if f is invertible, then the inverse is unique.

invertible → bijection
¬bijection → ¬invertible



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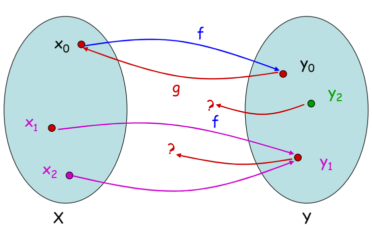

Not Bijection

Not Surjective: Something does not have an incoming arrow

OR

Not Injective: Something has more than one incoming arrow

In either case, we cannot construct an inverse

Back to combinatorics...

The Bijection Principle: If A and B are finite sets and there is a bijection from A to B, then |A| = |B|.

One way to count the number of elements in a set A is to show that there is a bijection from A to some other set B and count the number of elements in B.

How many subsets are there (including the empty set) of a set with n elements?

$$\sum_{k=0}^{n} C(n, k) = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

How many subsets are there (including the empty set) of a set with 4 elements?

$$\sum_{k=0}^{4} C(n, k) = 2^4$$

$$\sum_{k=0}^{4} \binom{n}{k} = 2^4$$

There is a one-to-one correspondence between subsets and binary numbers of length 4, and we know that there are 16 such numbers.

So by the Bijection Principle, there are 16 subsets.

n = 4

1	2	3	4
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
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Equivalent problem: How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 5 \quad \text{where } 0 \le x_i \le 5 \text{ ?}$$

There is a one-to-one correspondence between solutions to one problem and solutions to another.

Without answering either question, why do these two questions have the same answer?

- How many sets of size 2 can be made choosing elements from {1, 2, 3, 4, 5, 6, 7, 8, 9}?
- How many sets of size 7 can be made choosing elements from {1, 2, 3, 4, 5, 6, 7, 8, 9}?

Because every solution to one of the questions also specifies a solution to the other.

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