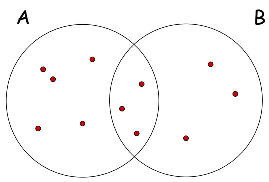
Sets

What is |A ∪ B| ? (A and B not necessarily disjoint)



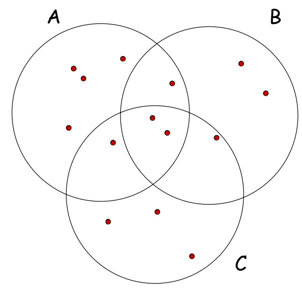
|A| = 8 |B| = 6

|A| + |B| = 14, which double counts the intersection

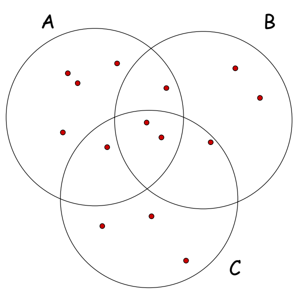
|A ∪ B| = |A| + |B| - |A ∩ B|

This is known as the Principle of Inclusion-Exclusion.

What is |A ∪ B ∪ C| ?

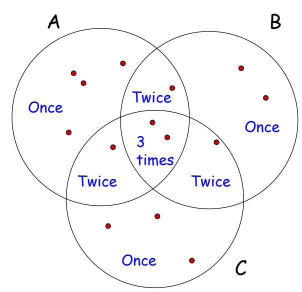


What is |A ∪ B ∪ C| ?



Counted by: |A| + |B| + |C|

What is |A ∪ B ∪ C| ?



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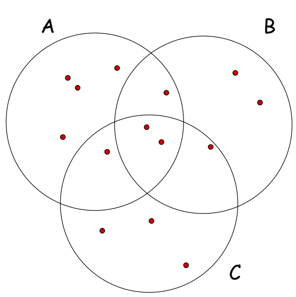

Counted by: |A| + |B| + |C| - |A∩B| - |A∩C| - |B∩C|

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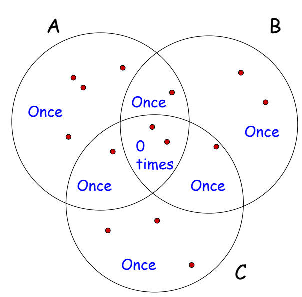

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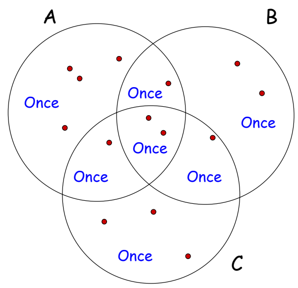
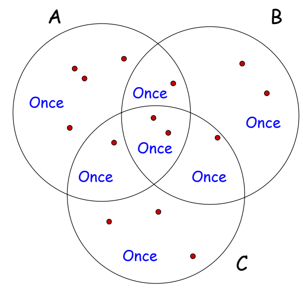


Counted by: $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

What is |A ∪ B ∪ C| ?



Counted by: $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
8 + 6 + 7 - 3 - 3 - 3 + 2
14

A survey of 200 TV viewers found that 110 watch sports, 120 watch comedy, 85 watch drama, 50 watch drama and sports, 70 watch comedy and sports, 55 watch comedy and drama, and 30 watch all three.

How many people watch sports, comedy, or drama?
How many do not watch any of these categories?

$|S \cup D \cup C| =$

$|S| + |D| + |C| - |S \cap D| - |S \cap C| - |D \cap C| + |S \cap D \cap C|$

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110 + 85 + 120 - 50 - 70 - 55 + 30

170 people watch sports, comedy, or drama
30 people do not watch any of these categories

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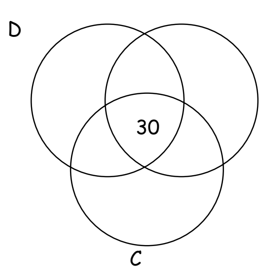

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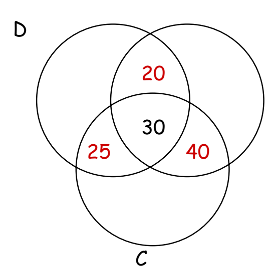

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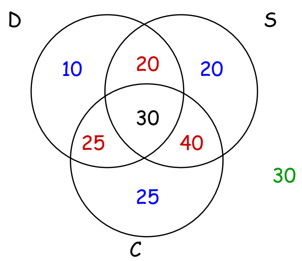
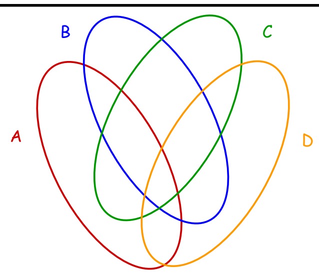


$|S \cup D \cup C| =$

$|S| + |D| + |C| - |S \cap D| - |S \cap C| - |D \cap C| + |S \cap D \cap C|$



$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$
$-|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$
$+|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$
$- |A \cap B \cap C \cap D|$

Permutations and Combinations

A **permutation** of a set of objects is an ordering of the objects.

For example, the set of elements {a b c} can be ordered in the following ways:

abc acb cba bac bca cab

By the product rule, there are $n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 2 \cdot 1 = n!$ permutations.

For any integer $n \geq 0$, the number of permutations of a set with n elements is n!

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1) \cdots 2 \cdot 1 & \text{if } n > 0 \end{cases}$$

How many ways can five people line up for tickets a concert?

r-permutations

An r-permutation is an ordering of r elements selected from a set of n elements.

{a, b, c}

The 2-permutations are:

ab ac ba bc ca cb

We use the notation P(n, r) to refer to the number of r-permutations of a set of n elements.

r-permutations

What is the value of P(n, r)? That is, how many ways can we order r elements from a set of n elements?

n ways to choose the 1st element
n-1 ways to choose the 2nd element
n-2 ways to choose the 3rd element
.
.
.
n-r+1 ways to choose the rth element

r-permutations

What is the value of P(n, r)? That is, how many ways can we order r elements from a set of n elements?

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n-r+1 ways to choose the rth element

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

r-permutations

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdots 1$$

r-permutations

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot \underbrace{(n-r) \cdot (n-r-1) \cdots 1}_{\text{Divide both sides by this}}$$

r-permutations

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot \underbrace{(n-r) \cdot (n-r-1) \cdots 1}_{\text{Divide both sides by this}}$$

This is P(n,r)

$$\frac{n!}{\underbrace{(n-r) \cdot (n-r-1) \cdots 1}_{\text{This is (n-r)!}}} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdots 1}{(n-r) \cdot (n-r-1) \cdots 1}$$

These cancel

$$\frac{n!}{(n-r)!} = P(n, r)$$

r-permutations

$$P(n, r) = \frac{n!}{(n-r)!} \quad \text{where } 0 \le r \le n$$

P(n, 0) = 1

P(0, 0) = 1

P(n, n) = n!

(recall that 0! is defined to be 1)

Suppose 5 members of a group of 12 are to be chosen to fill the following offices: president, vice-pres., secretary, treasurer, and parliamentarian.

How many different slates of officers could be chosen?

What fraction of solitaire hands initially show one or more Aces?

Possible hands =
Possible hands with no Aces =
Possible hands with one or more Aces =

Fraction of hands with one or more Aces =

What fraction of solitaire hands initially show one or more Aces?

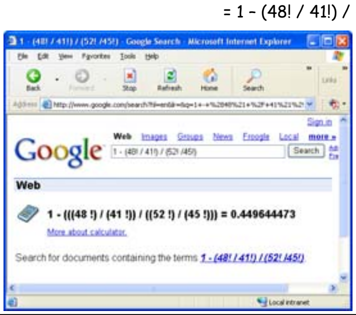
Possible hands = P(52, 7)
Possible hands with no Aces = P(48, 7)
Possible hands with one or more Aces = P(52, 7) – P(48, 7)

Fraction of hands with one or more Aces = $\frac{P(52, 7) - P(48, 7)}{P(52, 7)}$

Fraction of hands with one or more Aces = $\frac{P(52, 7) - P(48, 7)}{P(52, 7)}$

= 1 – P(48, 7) / P(52, 7)

= 1 – (48! / 41!) / (52! / 45!)



Combinations

Suppose 5 members of a group of 12 are to be chosen as a team to work on a project. How many distinct 5-person teams can be selected?

Or in general:

Given a set S with n elements, how many subsets of size r can be chosen from S?

Each individual subset of S of size r, is called an **r-combination** of S.

Since r-combinations are sets, they are unordered.

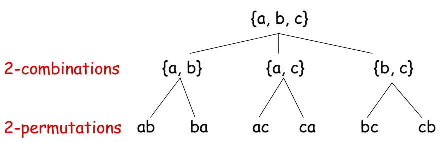
The 2-combinations of {a, b, c} are {a, b}, {a, c}, and {b, c}.

The number of subsets of size r than can be chosen from n elements

is denoted by $\binom{n}{r}$ or by C(n, r). We read these as "n choose r".

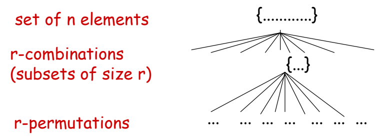
Combinations

We notice that there are more r-permutations of n elements than r-combinations. In fact, one way to get the permutations is to form the combinations, then form all the permutations of each of those:



Combinations

More generally:



We know that there are r! permutations of each set of r elements, so the total number of permutations at the bottom level, P(n, r), is given by

$$P(n, r) = C(n, r) \cdot r!$$

We can solve this for C(n, r)...

$$P(n, r) = C(n, r) \cdot r!$$

$$C(n, r) = \frac{P(n, r)}{r!}$$

Since we know that $P(n, r) = \frac{n!}{(n-r)!}$

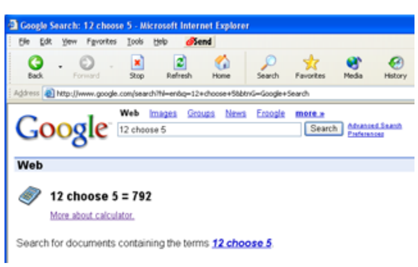
we have $C(n, r) = \frac{n!}{r!\,(n-r)!}$ for $0 \le r \le n$

$$C(n, r) = \frac{n!}{r!\,(n-r)!} \quad \text{for } 0 \le r \le n$$

$$C(n, 0) = C(n, n) = C(0, 0) = 1$$

$$C(12, 5) = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$C(12, 5) = 792$$



Suppose that we are choosing a team of 5 people from a group of 12, and that 2 people, Fred and Ginger, must either both be on the team or both be off the team. How many teams can we form?

teams with both + teams with neither

$C(10, 3) + C(10, 5)$

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$C(10, 3) + C(10, 5)$

Suppose Fred and Ginger refuse to work together. How many teams can we form?

Additive approach: teams with Fred + teams with Ginger + teams with neither

Subtractive approach: all possible teams - teams with both Fred and Ginger

Pascal's Identity

$C(n+1, k) = C(n, k-1) + C(n, k)$ for $n \geq k \geq 1$

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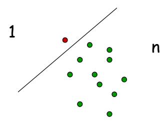

total number of subsets = number including • + number not including •

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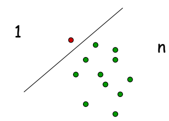

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$C(n, k-1) + C(n, k)$

Pascal's Identity

$C(n+1, k) = C(n, k-1) + C(n, k)$ for $n \geq k \geq 1$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \text{for } n > k \geq 1$$

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n ↓

C(0,0)

C(1,0) C(1,1)

C(2,0) C(2,1) C(2,2)

C(3,0) C(3,1) C(3,2) C(3,3)

...

k →

Pascal's Triangle

C(0,0)

C(1,0) C(1,1)

C(2,0) C(2,1) C(2,2)

C(3,0) C(3,1) C(3,2) C(3,3)

1 3 3 1

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

C(0,0)

C(1,0) C(1,1)

C(2,0) C(2,1) C(2,2)

C(3,0) C(3,1) C(3,2) C(3,3)

1 3 3 1

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

C(0,0)

C(1,0) C(1,1)

C(2,0) C(2,1) C(2,2)

C(3,0) C(3,1) C(3,2) C(3,3)

1 3 3 1

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

The Binomial Theorem:

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

C(0,0)

C(1,0) C(1,1)

C(2,0) C(2,1) C(2,2)

C(3,0) C(3,1) C(3,2) C(3,3)

C(4,0) C(4,1) C(4,2) C(4,3) C(4,4)

1 4 6 4 1

The Binomial Theorem:

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$\sum_{k=0}^{n} C(n, k) = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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The Binomial Theorem:

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

n = 4

```
0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
0 1 0 1
0 1 1 0
0 1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1
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