Combinatorics
The Art and Science of Counting

Even Hollywood is interested:

"There's only so many hands in a deck of cards."

From the movie Shane

Combinatorics
The Art and Science of Counting

Why count?

- To understand the performance of algorithms, we need to count the steps they execute
- We also need to count the amount of memory used as algorithms execute
- Counting is important in the study of probability, which is used in many algorithms and games
- Counting alternatives is often important in algorithm design

Suppose there are 18 math majors and 200 CS majors at Stanford.

How many ways are there to pick one representative who is either a math major or a CS major?

> The Sum Rule: If a task can be accomplished by choosing one of the n_A alternatives in set A or by choosing one of the n_B alternatives in set B, and if the sets A and B are disjoint, then there are $n_A + n_B$ ways to accomplish the task. This can be generalized to any number of tasks.

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How many ways are there to pick two representatives, so that one is a math major and one is a CS major?

> The Product Rule: If a task consists of a sequence of two subtasks, and there are n_1 ways to accomplish the first subtask, and for each of these there are n_2 ways to accomplish the second subtask, then there are $n_1 n_2$ ways to accomplish the overall task. This can be generalized to any number of tasks.

Suppose there are 18 math majors and 200 CS majors at Stanford.

How many ways are there to pick one representative who is either a math major or a CS major?

How many ways are there to pick two representatives, so that one is a math major and one is a CS major?

How many ways are there to pick two representatives, regardless of their majors?

> The Sum Rule: If a task can be accomplished by choosing one of the n_A alternatives in set A or by choosing one of the n_B alternatives in set B, and if the sets A and B are disjoint, then there are $n_A + n_B$ ways to accomplish the task. This can be generalized to any number of tasks.

Suppose you are either going to go to

an Italian restaurant that serves 15 entrées,

or to

a French restaurant that serves 10 entrées.

How many choices for an entrée do you have?

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Suppose you go to a French restaurant and find out that the *prix fixe* menu is three courses, with a choice of 4 appetizers, the 10 entrées, and 5 desserts.

How many different meals can you have?

How many different three-letter uppercase initials are there (with repetition and without)?

How many different three-letter uppercase initials are there that begin with the letter A?

How many binary numbers of length 8 begin and end with a 1?

How many strings of lowercase letters are there of length four or less?

Suppose you have 5 computer science books, 3 math books, and 2 art books. How many ways can you select two books from different subjects?

How many binary numbers of length eight either start with a 1 or end with 00?

Sets

A set is an unordered collection of distinct objects, which we call the elements of the set.

The set of no elements is called the empty set.

If A is a finite set, $|A|$ denotes the number of elements in A, which is called the cardinality of A.

The union of sets A and B, denoted $A \cup B$, is the set of all elements in A or B.

The intersection of sets A and B, denoted $A \cap B$, is the set of all elements in both A and B.

Generalized Sum and Product Rules

The Sum Rule: If a task can be accomplished by choosing one of the alternatives from the sets $S_1, S_2, \ldots, S_m$, and these sets are pairwise disjoint (i.e., $S_i \cap S_j = \emptyset$ for all $i \neq j$), and n_i is the number of elements in S_i, then the number of ways to accomplish the task is $n_1 + n_2 + \ldots + n_m$.

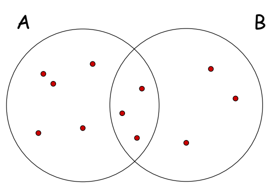
Using the notation of set theory, we would write

$|S_1 \cup S_2 \cup \ldots \cup S_m| = |S_1| + |S_2| + \ldots + |S_m|$ (where the sets are disjoint).

The Product Rule: If $E_1, E_2, \ldots, E_m$ is a sequence of events such that E_1 can occur in n_1 ways and if $E_1, E_2, \ldots, E_{k-1}$ have occurred, then E_k can occur in n_k ways, then there are $n_1 n_2 \ldots n_m$ ways in which the entire sequence of events can occur.

Sets

What is $|A \cup B|$? (A and B not necessarily disjoint)



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$|A| + |B| = 14$, which double counts the intersection

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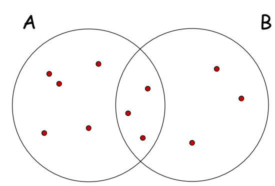

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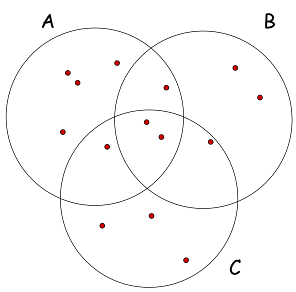
$|A \cup B| = |A| + |B| - |A \cap B|$

This is known as the Principle of Inclusion-Exclusion.

What is $|A \cup B \cup C|$?



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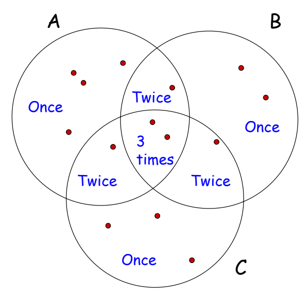

Counted by: $|A| + |B| + |C|$

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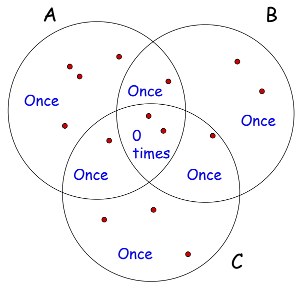

Counted by: $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$

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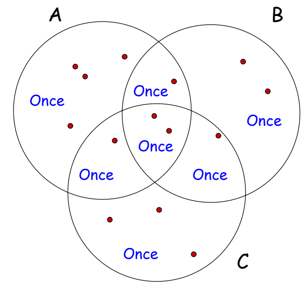

Counted by: $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

8 + 6 + 7 - 3 - 3 - 3 + 2

14

A survey of 200 TV viewers found that 110 watch sports, 120 watch comedy, 85 watch drama, 50 watch drama and sports, 70 watch comedy and sports, 55 watch comedy and drama, and 30 watch all three.

How many people watch sports, comedy, or drama?
How many do not watch any of these categories?

$|S \cup D \cup C|$ =

$|S| + |D| + |C| - |S \cap D| - |S \cap C| - |D \cap C| + |S \cap D \cap C|$

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170 people watch sports, comedy, or drama
30 people do not watch any of these categories

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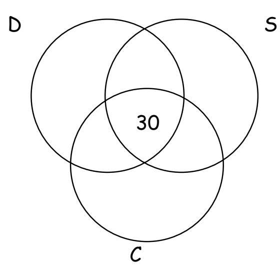

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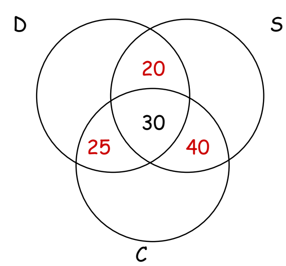

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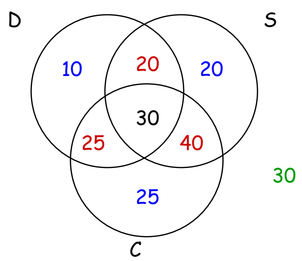

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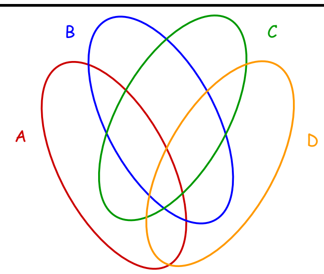
$|S| + |D| + |C| - |S \cap D| - |S \cap C| - |D \cap C| + |S \cap D \cap C|$



$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$
$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$
$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$
$- |A \cap B \cap C \cap D|$

Permutations and Combinations

A **permutation** of a set of objects is an ordering of the objects.

For example, the set of elements {a b c} can be ordered in the following ways:

abc acb cba bac bca cab

By the product rule, there are $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!$ permutations.

For any integer $n \geq 0$, the number of permutations of a set with n elements is n!

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1) \cdots 2 \cdot 1 & \text{if } n > 0 \end{cases}$$

How many ways can five people line up for tickets a concert?

r-permutations

An r-permutation is an ordering of r elements selected from a set of n elements.

{a, b, c}

The 2-permutations are:

ab ac ba bc ca cb

We use the notation P(n, r) to refer to the number of r-permutations of a set of n elements.

r-permutations

What is the value of P(n, r)? That is, how many ways can we order r elements from a set of n elements?

n ways to choose the 1st element
n-1 ways to choose the 2nd element
n-2 ways to choose the 3rd element
.
.
.
n-r+1 ways to choose the r^{th} element

$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$

r-permutations

$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$

$n! = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdots 1$

r-permutations

$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$

$n! = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot \underbrace{(n-r) \cdot (n-r-1) \cdots 1}_{\text{Divide both sides by this}}$

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This is P(n,r)

$$\frac{n!}{\underbrace{(n-r) \cdot (n-r-1) \cdots 1}_{\text{This is (n-r)!}}} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdots 1}{(n-r) \cdot (n-r-1) \cdots 1}$$

These cancel

$$\frac{n!}{(n-r)!} = P(n, r)$$

r-permutations

$$P(n, r) = \frac{n!}{(n-r)!} \quad \text{where } 0 \le r \le n$$

$P(n, 0) = 1$

$P(0, 0) = 1$

$P(n, n) = n!$