Handout #50 | CS103A
November 10, 2008 | Robert Plummer

Problem Set #9

Due: November 19

Part I: 1 point each

1) Give a recursive definition of the sequence $a_1, a_2, \ldots a_n$ when the n^{th} element can be found using the formula $a_n = 6n$.

2) Prove by induction that for $x \neq 0$, $\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$ for $n \geq 0$

Part II: 3 points each

3) In class we determined that the number of moves needed to solve the Tower of Hanoi puzzle with n disks is given by

$H_1 = 1$
$H_n = 2H_{n-1} + 1$ for $n > 1$

Guess a non-recursive (explicit) formula for H_n and prove your formula is correct.

The next three problems involve writing simple recursive algorithms in pseudocode. Here is an example of the style we are looking for:

The sum of the first n positive integers is given by Sum(n) : if n = 0 then return(0) else return(n + Sum(n-1))

Your algorithm should be a single "statement" of the form
if *condition* then *statement* else *statement*

Notice that the definition of "statement" is itself recursive, so that you can have something like this:
if ... then ... else if ... then ... else if ... then ... else ...
Another implication is that the only variable is the argument to the function (like n in the example above). You don't have a way to define additional variables, nor will you need them for these exercises.

4) Give a recursive algorithm for the function Add(a, b) that returns a + b, where a and b are non-negative integers. Assume that the only arithmetic computations that you can carry out are adding 1 to a number and subtracting 1 from a number.

5) This problem concerns strings of digits, like "6743832453". You are given the following functions, where S is a string of digits:

Length(S) = the number of digits in S (so Length("6743832453") = 10)
First(S) = the first digit in S (so First("6743832453") = 6)
Rest(S) = S with its first digit removed (so Rest("6743832453") = "743832453")

We use Ø to stand for the "empty string" of no digits, so Length(Ø) = 0, Rest("3") = Ø, First(Ø) is undefined, and Rest(Ø) is undefined. Note also that we distinguish between strings and digits. E.g. the argument to Length must be a string, and the value returned by First is a digit. This means that a string of length 1 is not the same thing as the digit that comprises it; e.g. "3" is a string, but 3 is a digit.

Give a recursive algorithm for the function Largest(S) that finds the largest digit that appears in the string S. For example, Largest("6743832453") = 8. You may assume that the input to Largest is not the empty string.

(more problems on the back)

6) In this problem you may use the same three functions that you were given in problem 5, and the additional function Cons(d, S) that takes a digit and a string as arguments and returns a new string that is S with d added to the front. So Cons(4, "386") = "4386", and Cons(2, Ø) = "2". (Note: the name "Cons" comes from "construct".)

Give a recursive algorithm for the function Concat(S1, S2) that returns the concatenation of the two strings that are its arguments. Here are some examples:

Concat("976", '23") = "97623"
Concat("1", "45") = "145"
Concat("231", "6") = "2316"
Concat("37", Ø) = "37"
Concat(Ø, "428") = "428"
Concat(Ø, Ø) = Ø

7) Let S be the set of ordered pairs of integers defined recursively by:

Base case: $(0,0) \in S$
Recursive Step: If $(a,b) \in S$ then $(a+2, b+3) \in S$ and $(a+3, b+2) \in S$

Prove using induction: $5 \mid (a+b)$ when $(a,b) \in S$

One thing you will have to decide is what to do the induction on.

8) Suppose that m is a positive integer and F is the Fibonacci sequence. Show that for all positive integers n,

$F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$

Your answer will be a proof by induction on n.

9) Give a recursive definition for the set of all strings of balanced parentheses. The empty string is not a member of this set. Examples of strings that should be in the set are:

( )
((( )))
(( )) ( )

Note that a string like)(should not be in the set.

10) Give a recursive definition for the set of all strings of 1's and 0's that are palindromes. These are strings that read the same forward as backward, such as 0, 0110, and 1101011. The empty string is also a member of this set.

11) Suppose $a_1 = 3$, and $a_n = a_{n-1} + 2n$ for $n > 1$. Show that $a_n = n^2 + n + 1$.