CS103A HO# 47 11/7/08
Slides--Induction III, Recursion

CI → WOP

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Proof. Suppose that A is a set of positive integers without a least element, and let P(n) be the proposition that $n \notin A$. We will show that $\forall n\, P(n)$ by complete mathematical induction, i.e., that A must be empty.

BASE CASE: Since 1 is the smallest positive integer, $1 \notin A$, because if so, 1 would be the least element of A. So P(1) is true.

INDUCTIVE STEP:
Assume: for some positive integer k, P(i) for $1 \le i \le k$.
Show P(k + 1).

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Show P(k + 1).

If $k + 1 \in A$, $k + 1$ would be the least element in A, since no integer less than $k + 1$ is in A by the inductive hypothesis. Thus $k + 1 \notin A$, and P(k+1) is true.

Thus by C.I., $\forall n\, P(n)$. Since A is a set of positive integers to which no integer belongs, we have shown that if A has no least element, it must be empty.

This also proves the contrapositive: if A is not empty, it has a least element. ■

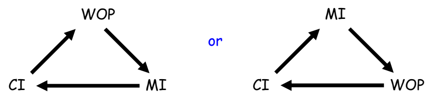
We can show that:

MI ↔ WOP

MI ↔ CI

CI ↔ WOP

Rather than six proofs, we would do three:



From last time: Bad Induction

Show that for any positive integer n, $n^3 - n + 1$ is a multiple of 3.

P(n): $n^3 - n + 1$ is a multiple of 3

We will show that if we assume P(k) is true, then P(k + 1) is true

$$(k + 1)^3 - (k + 1) + 1 = k^3 + 3k^2 + 3k + 1 - k = \underbrace{k^3 - k + 1}_{\text{Divisible by 3 by inductive hypothesis}} + \underbrace{3(k^2 + k)}_{\text{Obviously divisible by 3}}$$

So P(k + 1) is true and P(n) is true for all n.

What's wrong?

There is no base case. Not only is P(1) not true, P(n) is false for all n.

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We observe that P(1) is false. Thus $\exists n\, \neg P(n)$, and $\neg\forall n\, P(n)$.

But does this show $\forall n\, \neg P(n)$?

Does $(\neg P(1) \wedge (P(1) \rightarrow P(2))) \rightarrow \neg P(2)$?

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Proof: $n^3 - n + 1 = n\,(n^2 - 1) + 1$
$= \underline{n}\,\underline{(n + 1)}\,\underline{(n - 1)} + 1$

These are three consecutive integers, so one of them is divisible by 3.

Thus $(n^3 - n + 1) \bmod 3$ is always 1, and $\forall n \neg P(n)$.

Recursion

Specifying a Sequence

Enumerate: 2, 4, 6, 8, ...

Explicit formula: $a_n = 2n$, $n = 1, 2, \ldots$

Recursive formula: $a_1 = 2$, $a_n = a_{n-1} + 2$

Recursive Definition of a Sequence

1,	3,	6,	10,	15,	21,	...
a_1	a_2	a_3	a_4	a_5	a_6	

$a_1 = 1$
$a_n = a_{n-1} + n$

Solving the recurrence relation to obtain an explicit formula can be difficult, but here, we can see that a_i is the sum of the 1st n integers, so

$$a_n = \frac{n(n + 1)}{2}$$

which we proved by induction.

Recursive Definition of Factorial

If $n! = \begin{cases} 1 & \text{if } n = 1 \\ n(n-1)! & \text{if } n > 1 \end{cases}$

then $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

Base Case: 1! = 1 by both definitions

Inductive Step:
Assume P(k): k! defined recursively $= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k$
Show P(k+1): (k+1) defined recursively $= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (k+1)$

By recursive def. $(k+1)! = (k+1)(k+1-1)!$
$= (k+1)k!$
$= k!(k+1)$
$= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k \cdot (k+1)$ by Inductive Hypothesis

QED.

Fibonacci Sequence

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

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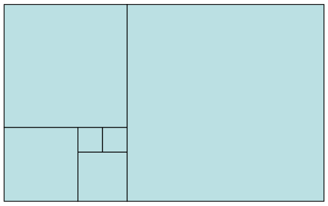
The Spiral



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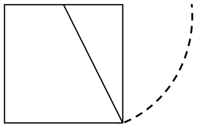
The Ratio



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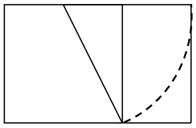

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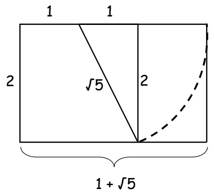
The Ratio

$$\frac{1 + \sqrt{5}}{2}$$

1.618034...



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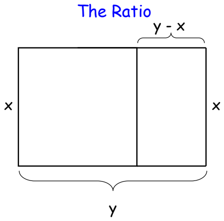

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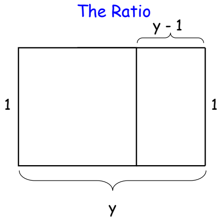

$$\frac{y-1}{1} = \frac{1}{y}$$

$$y^2 - y - 1 = 0$$

$$\frac{1 \pm \sqrt{5}}{2}$$

$$y - 1 = \frac{1}{y}$$

$$y = 1 + \frac{1}{y}$$

Fibonacci Sequence

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The Explicit Formula

$$F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

Fibonacci Sequence

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

Prove P(n) : $\sum_{i=1}^{n} F_i^2 = F_n \cdot F_{n+1}$ for $n \geq 1$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

BASE CASE: P(1) asserts that $1^2 = 1 \cdot 1$, which is true.

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Prove P(n) : $\sum_{i=1}^{n} F_i^2 = F_n \cdot F_{n+1}$ for $n \geq 1$

INDUCTIVE STEP: Assume P(k) : $\sum_{i=1}^{k} F_i^2 = F_k \cdot F_{k+1}$ for some $k \geq 1$

Show P(k+1): $\sum_{i=1}^{k+1} F_i^2 = F_{k+1} \cdot F_{k+2}$

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Show P(k+1): $\sum_{i=1}^{k+1} F_i^2 = F_{k+1} \cdot F_{k+2}$

$\sum_{i=1}^{k} F_i^2 + F_{k+1}^2 = F_k \cdot F_{k+1} + F_{k+1}^2$ by Inductive Hypothesis

$\sum_{i=1}^{k+1} F_i^2 = F_{k+1} \cdot (F_k + F_{k+1})$ def. of Σ, algebra

$= F_{k+1} \cdot F_{k+2}$ def. of Fibonacci Seq.

Fibonacci Sequence

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

Prove P(n) : $F_n < 2^n$ for $n \geq 0$

Base Case: $F_0 = 0 < 2^0$ so P(0) is true.
$F_1 = 1 < 2^1$ so P(1) is true.

Inductive Step:
Assume P(i) : $F_i < 2^i$ for $1 \leq i \leq k$ for some $k \geq 1$
Show P(k+1) : $F_{k+1} < 2^{k+1}$

$F_{k+1} = F_k + F_{k-1}$ for $k \geq 1$ by Def. Fib.
$F_{k+1} < 2^k + 2^{k-1}$ by Ind. Hyp.
$F_{k+1} < 2^k + 2^k$
$F_{k+1} < 2^{k+1}$