Handout #41 CS103A
November 3, 2008 Robert Plummer

Sequences and Summations

A mathematician, like a poet or a painter, is a maker of patterns.

G.H. Hardy
A Mathematician's Apology (1940)

Sequences

Imagine a person (with a lot of spare time) who decides to count her ancestors. She has two parents, four grandparents, eight great-grandparents, and so forth. We could write these numbers in a row: 2, 4, 8, 16, 32, 64, … (where the … means and so forth).

To express a pattern of numbers in this manner, we often label the position of each number in the row as in the following table.

1	2	3	4	5	6
2	4	8	16	32	64

When represented in this manner it is easy to recognize a formula that would give us the kth element in the row: $A_k = 2^k$. Note that we are just making an observation based on evidence in guessing this formula. We would need to do a proof to be absolutely certain.

> A <u>sequence</u> is an ordered list of elements written in a row, such that each element has a unique position in the list. We use a_k to denote a single element of a sequence called a <u>term</u>. The k in a_k is called a <u>subscript</u> or <u>index</u>. An <u>explicit formula</u> for a sequence is a rule that shows how the value of a_k is derived from k.

`What is the programming analogy of a sequence?`

A common problem in computer science is determining an explicit formula given only the first few elements of a sequence. When trying to find such a formula we try to find a pattern. A good place to start is in asking the following questions:

- Are there runs of the same values?
- Are terms obtained from previous terms by adding the same amount, or an amount that depends on position in the sequence?
- Are terms obtained from previous terms by multiplying by a particular amount?
- Are terms obtained by combining previous terms in a certain way?

Here are some practice problems:

```
7, 11, 15, 19, 23,  27,  31,   35...
3,  6, 11, 18, 27,  38,  51,   66,   83...
0,  2,  8  26, 80, 242, 728, 2186, 6560, 19682...
```

And just for fun:

```
O,  T,  T,  F,  F,  S,  S,  E, ...
```

Summation & Product Notation

Going back to our original question of counting ancestors, suppose we want to know the total number of ancestors for the past six generations. There is a convenient shorthand notation to write such sums. In 1772, the French mathematician Joseph Lagrange introduced the capital Greek letter sigma, $\sum$ to denote the word "sum".

$\sum_{k=1}^{n} a_k$ represents $a_1 + a_2 + a_3 + \ldots + a_n$

k is called the <u>index of summation</u> which starts with the <u>lower limit</u> 1, and ends with the <u>upper limit</u> n.

```
What is the value of the following?
```

$$\sum_{j=1}^{6} 2^j$$

```
What is the value of the following?
```

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij$$

The notation for the product of a sequence of numbers is analogous to the notation for their sum, except we use the capital Greek letter pi, $\prod$ to denote a product:

$\prod_{k=1}^{n} a_k$ represents $a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$ (using $\cdot$ for multiplication)

Arithmetic and Geometric Progressions

An arithmetic progression is a sequence of the form

$a, a+d, a+2d, \ldots, a+(n-1)d, \ldots$

where the *initial term* a and the *common difference* d are real numbers. Another way to write this is

$a + (n-1)d, \quad n = 1, 2, \ldots$

If we call the terms of this progression $t_1, t_2, \ldots$, then

$t_1 = a$

$t_n = a + (n-1)d$

The sum of an initial segment of an arithmetic progression is called an arithmetic series. A common problem is to find the sum of the first n terms of an arithmetic progression; i.e., of the series: $t_1 + t_2 + t_3 + \ldots + t_{n-2} + t_{n-1} + t_n$

You are probably familiar with the formulas for this sum, based on the fact that $t_n = a + (n-1)d$:

$$\sum_{k=1}^{n} t_k = \frac{n(2a + (n-1)d)}{2} = \frac{n(t_1 + t_n)}{2}$$

A familiar case of this sum is for a = 1 and d = 1. The sequence is 1, 2, 3, 4, 5, ... and we have

$$\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

How are these formulas derived?

A geometric progression is a sequence of the form

$a, ar, ar^2, ar^3, \ldots, ar^n$

where a, the *initial term* and r, the *common ratio* are real numbers. It is common to index the terms in a geometric progression starting with index 0. If we call the terms of this progression $t_0, t_1, \ldots$, then we have

$t_0 = a$ (the first term)

$t_n = ar^n$ (the n+1st term)

Sums of geometric progressions are very common in discrete math and computer science. Such sums are called geometric series. There is a well-known and useful formula for the sum, S_n, of the first n + 1 terms of a geometric progression. Let's first consider the case where a = 1, $r \neq 1$:

$$S_n = \sum_{k=0}^{n} t_k = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r} = \frac{r^{n+1} - 1}{r - 1}$$

```
How is this formula derived?
```

If we take the more general case where a could have a value other than 1, we have

$$S_n = \sum_{k=0}^{n} t_k = \sum_{k=0}^{n} ar^k = a\sum_{k=0}^{n} r^k = \frac{a(1 - r^{n+1})}{1 - r} = \frac{a(r^{n+1} - 1)}{r - 1} \qquad \text{where } r \neq 1$$

```
Prove or disprove: When a constant amount is added to each term of a
geometric sequence, it is no longer a geometric sequence.
```

Approach 1: Consider a geometric sequence S = { a, ar, ar^2, ar^3, …, ar^n}. Add a constant "b" to each element giving the sequence S' = { a + b, ar + b, ar^2 + b, ar^3 + b, …, ar^n + b}. Is S' a geometric sequence?

To be a geometric sequence we know

$$\frac{s_{n+1}}{s_n} = \frac{s_n}{s_{n-1}} = r \qquad \text{(why?)}$$

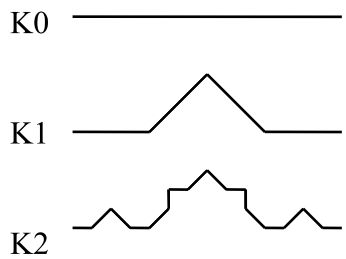
```
How would you complete this proof?
```

Approach 2: Can you come up with a counterexample?

```
Prove or disprove: When a constant amount is multiplied to each term of a
geometric sequence, it is no longer a geometric sequence.
```

Fractals (for general interest--not considered part of course material)

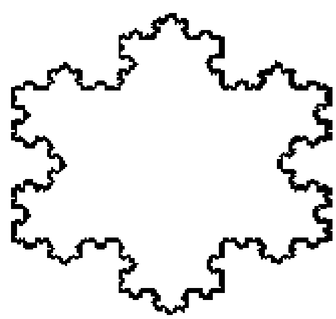
Consider the following curve:



This is the famous *Koch curve*, first generated in 1904 by the Swedish mathematician Helge von Koch. It stirred a lot of interest because it produces an infinitely long line in a finite area. We begin with a horizontal line of say, length 1. To create a first-order Koch curve, we divide the line in thirds and replace the middle section with a triangular bump having sides of 1/3. The total line length is evidently 4/3. The second-order Koch curve is generated by building a bump on each of the four sides of a first-order Koch curve. Because each segment is increased in length by a factor of 4/3, the total curve length is 4/3 larger. In general, we form a K_{i+1} Koch curve from K_i by placing a bump on each of its line segments. As i tends toward infinity, the length of the curve also appears to become infinite, yet the curve is in a finite area.

`Can the number of line segments at each level of a Koch curve be represented by a geometric sequence?`

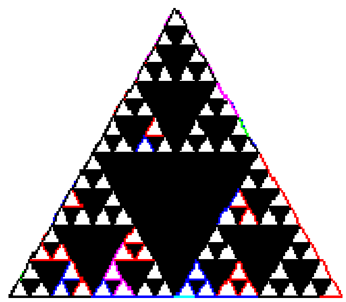
"Koch-ing" the three sides of an equilateral triangle with fourth-order Koch curves gives us the famous Koch snowflake.



The Koch curve (and there are many others) have been refined by replacing a figure with similar but smaller versions of itself. This leads to the idea of *self-similarity* in a shape. The level of detail remains the same no matter how closely one looks at the figure. For example, consider a Koch curve taken to its limit ($K\infty$), one could magnify a part of the picture a billion times and the basic details would be the same.

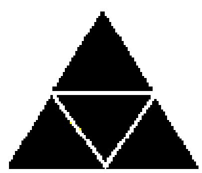
Nature provides many examples that mimic self-similarity. The classic example is a coastline. Seen from a satellite, it has a certain level of ruggedness caused by bays, inlets and peninsulas. Zoom in to a view from an airliner and we perhaps see the details of a particular bay. But the bay has a ruggedness of its own with individual boulders and undulations in the beach. Zoom in to walking on that beach and smaller rocks and pebbles seem to produce the same level of ruggedness. There are many other natural examples of self-similarity, e.g., the cellular structure of a leaf, the blood vessel system in animals, etc.

Here is another famous fractal: the Sierpinski triangle



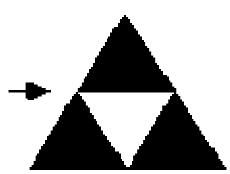
Step One

Draw an equilateral triangle with sides of 2 triangle lengths each. Connect the midpoints of each side.



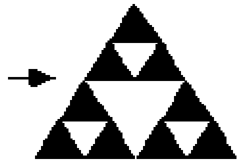
How many equilateral triangles do you now have?

Shade out the triangle in the center. Think of this as cutting a hole in the triangle.



Step Two

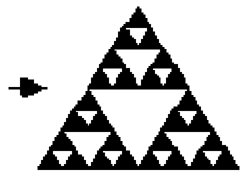
Draw another equilateral triangle with sides of 4 triangle lengths each. Connect the midpoints of the sides and shade the triangle in the center as before.



Notice the three small triangles that also need to be shaded out in each of the three triangles on each corner - three more holes.

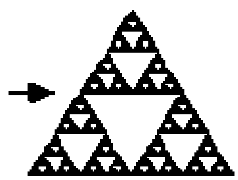
Step Three

Draw an equilateral triangle with sides of 8 triangle lengths each. Follow the same procedure as before, making sure to follow the shading pattern. You will have 1 large, 3 medium, and 9 small triangles shaded.



Step Four

Follow the above pattern and complete the Sierpinski Triangle.



The number of triangles created at each level can be represented by a geometric sequence.

```
How many new triangles are produced at each level?  How many of these new
triangles will be subdivided at the next level?
```

Original handout by Maggie Johnson, modified by Robert Plummer