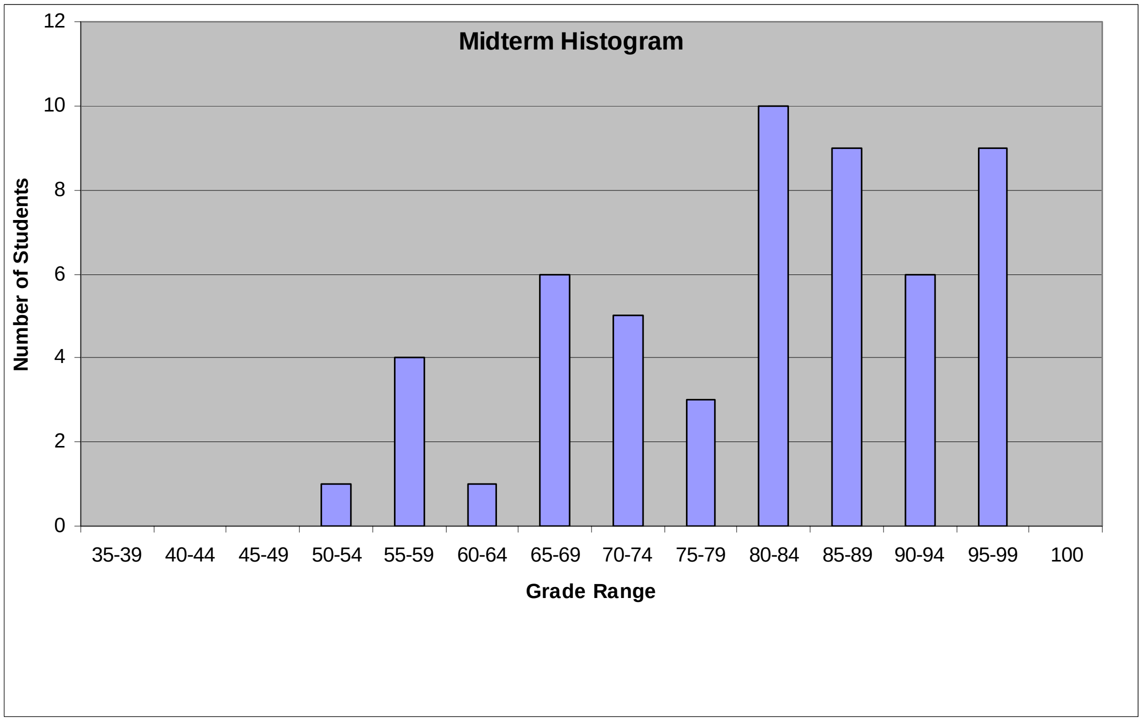
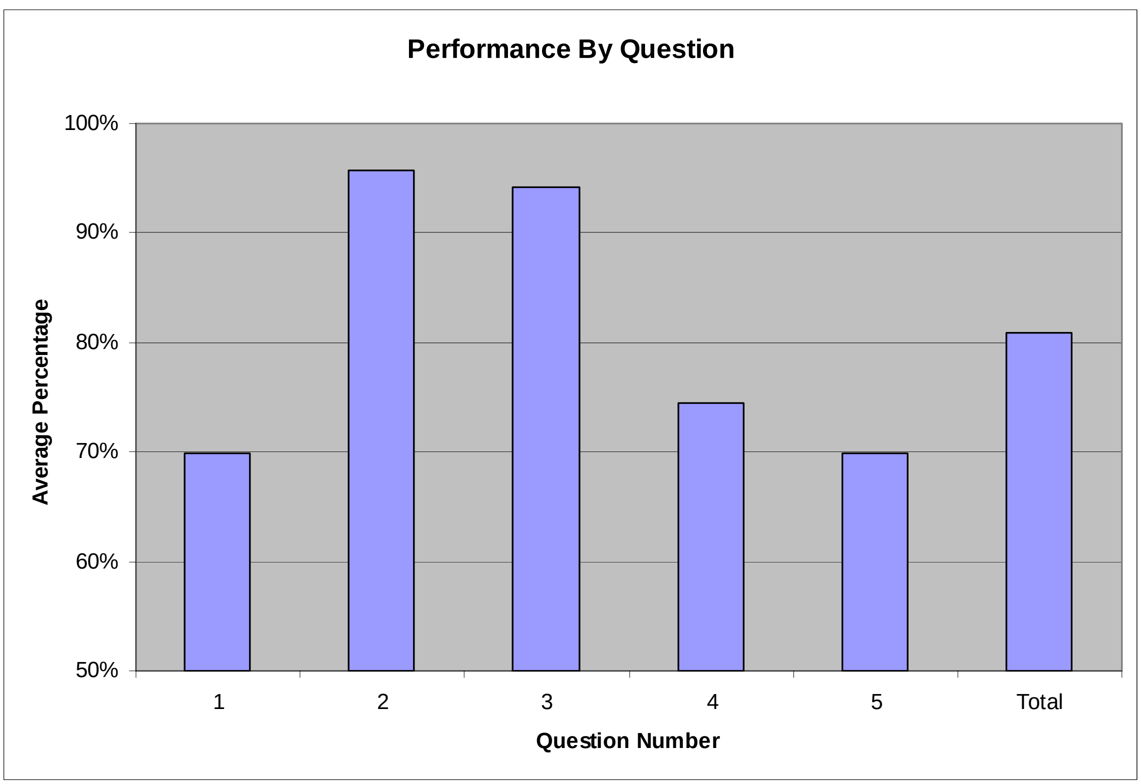
Handout #38 CS103A
October 29, 2008 Robert Plummer

Midterm Exam Solutions

The average on the midterm was 80.0.





1. Tautology, First-Order Validity, and Logical Truth (20 points)

For each of the following sentences, circle the **correct** description:

(1) $((a = b) \wedge (b = c)) \rightarrow (a = c)$

(a) Tautology
(b) First-Order Valid, but not tautology
(c) Logical Truth, but not First-Order valid
(d) Not Logical Truth

Since the meaning of = is important, this is F.O. Valid, not a Tautology.

(2) $((\forall x Cube(x) \vee \forall x Tet(x)) \wedge (\neg\forall x Tet(x) \vee \exists x Dodec(x))) \rightarrow (\forall x Cube(x) \vee \exists x Dodec(x))$

(a) Tautology
(b) First-Order Valid, but not tautology
(c) Logical Truth, but not First-Order valid
(d) Not Logical Truth

(3) $\neg\exists x\exists y(LeftOf(x, y) \wedge \neg Cube(x) \wedge Tet(y)) \leftrightarrow \forall x\forall y(Tet(y) \rightarrow (LeftOf(x, y) \rightarrow Cube(x)))$

(a) Tautology
(b) First-Order Valid, but not tautology
(c) Logical Truth, but not First-Order valid
(d) Not Logical Truth

(4) $\forall x\forall y(\neg LeftOf(x, y) \rightarrow (RightOf(x, y) \vee x = y))$

(a) Tautology
(b) First-Order Valid, but not tautology
(c) Logical Truth, but not First-Order valid
(d) Not logical truth

The sentence would be false in a world where a and b are two blocks that are in the same column.

2. Equivalences (20 points)

For each given sentence, circle the choice that is a First-Order Equivalence.

(1) $\mathbf{(P \vee Q) \vee (P \rightarrow Q)}$

a) $(Q \vee P) \vee (Q \rightarrow P)$

b) $P \wedge \neg P \wedge Q$

c) $\mathbf{P \vee \neg P \vee Q}$

d) $\mathbf{P \vee (P \rightarrow Q)}$ (Two answers were correct on this one!)

(2) $\mathbf{\neg (P \vee (Q \rightarrow R))}$

a) $R \rightarrow (Q \vee P)$

b) $\mathbf{\neg P \wedge \neg R \wedge Q}$

c) $\neg P \wedge (\neg R \vee Q)$

d) $P \wedge (\neg R \vee \neg Q)$

(3) $\mathbf{\forall x\, (P(x) \rightarrow Q(x))}$

a) $\neg\exists x\, \neg(\neg Q(x) \rightarrow P(x))$

b) $\mathbf{\neg\exists x\, \neg(\neg Q(x) \rightarrow \neg P(x))}$

c) $\neg\exists x\, (\neg Q(x) \rightarrow \neg P(x))$

d) $\neg\exists x\, \neg(\neg Q(x) \vee \neg P(x))$

(4) $\mathbf{\forall x\, \forall y\, (R(x) \wedge Q(x, y)) \wedge \neg\exists z\, \neg R(z)}$

a) $\mathbf{\forall x\, \forall y\, (Q(x,y) \wedge R(x))}$

b) $\mathbf{\forall x\, \forall y\, (Q(y,x) \wedge R(x))}$

c) $\forall x\, \forall y\, \forall z\, (Q(x, y) \wedge R(x,z))$

d) $\mathbf{\forall x\, \forall y\, \forall z\, (Q(x, y) \wedge (R(x) \vee R(z)))}$

(OK, this question really had problems! It says that Q is true for every pair of objects, and that R is true for everything. The three answers indicated are all correct.)

3. Conditionals (20 points)

Prove or disprove the following. If you prove it, write up a Fitch-style proof with <u>all</u> steps included. Number your steps and refer to those numbers in the reasons you give for each step. If the argument is not valid, describe in detail where the argument fails.

You may **not** use Taut Con in your proof except for the law of excluded middle. Note: the fact that there is nothing above the Fitch bar indicates that there are no premises for this proof.

1.		
2.	$A \rightarrow (C \rightarrow \neg B)$	
3.	$A \rightarrow B$	
4.	A	
5.	B	→ Elim: 4,3
6.	$C \rightarrow \neg B$	→ Elim: 4,2
7.	C	
8.	$\neg B$	→ Elim: 6,7
9.	$\bot$	⊥ Intro: 5,8
10.	$\neg C$	¬ Intro: 7-9
11.	$A \rightarrow \neg C$	→ Intro: 4-10
12.	$(A \rightarrow B) \rightarrow (A \rightarrow \neg C)$	→ Intro: 3-11
13.	$(A \rightarrow (C \rightarrow \neg B)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow \neg C))$	→ Intro: 2-12

4. Translations (20 points)

Translate each of the following sentences into first-order logic. The statements describe situations involving Toddlers, Nannies, and Blocks. Billy is a constant referring to a particular toddler. You may only use the following predicates:

IsSuper(x, y)	x is supervising y
Toddler(x)	x is a toddler
Sleeping(x)	x is sleeping
Nanny(x)	x is a nanny
Block(x)	x is a block
FightsWith(x, y, z)	x fights with y over z

(1) **Nobody who is supervising can be sleeping or fighting.**

$$\forall x \forall y \forall z \forall w \left[IsSuper(x, y) \rightarrow [\neg Sleeping(x) \wedge \neg FightsWith(x, z, w)] \right]$$

Here is an alternate:

$$\forall x(\exists y IsSuper(x, y) \rightarrow \neg[Sleeping(x) \vee \exists w \exists z Fighting(x, z, w)])$$

(2) **Some nanny is supervising (at least) two toddlers**.

$$\exists x \exists y \exists z[Nanny\ (x) \wedge Toddler\ \ (y) \wedge Toddler\ \ (z) \wedge\ y \neq\ z \wedge IsSuper\ \ (x, y) \wedge IsSuper\ \ (x, z)]$$

(3) **Billy fights with another toddler over a block.**

$$\exists x \exists y[FightsWith(Billy, x, y) \wedge Toddler(x) \wedge Billy \neq x \wedge Block(y)]$$

(4) **Every toddler who is sleeping is being supervised by some nanny—or it is the case that all nannies are sleeping.**

$$\forall x \big((Toddler(x) \wedge Sleeping(x)) \rightarrow \\ \exists w(Nanny(w) \wedge IsSuper(w, x)) \big) \vee \forall v(Nanny(v) \rightarrow Sleeping(v))$$

(5) **The only sleeping nanny is the one supervising Billy.**

$$\exists x \forall y \big(Sleeping(x) \wedge Nanny(x) \wedge IsSuper(x, Billy) \wedge \\ [(Sleeping(y) \wedge Nanny(y)) \rightarrow x = y] \big)$$

5. Formal Proofs with Quantifiers (20 points)

Give a Fitch-style proof for the following. Number your steps and refer to those numbers in the reasons you give for each step. You **are allowed to use Taut Con freely** to justify proof steps where it would be accepted by the program Fitch.
Hint: only one boxed constant is needed for this proof.

This proof uses the Law of Excluded Middle. There is another proof on the next page.

```
1. ∀y(Small(y) → Plant(y))
2. ∃x ¬Small(x) → ¬∃x Animal(x)
3. ∀x ∀y [(¬Animal(x) ∧ ¬Plant(x)) → Small(y)]
   ---
   | 4. [c]
   | ---
   | 5. Small(c) ∨ ¬Small(c)                          Taut Con:
   |   | 6. Small(c)
   |   | ---
   |   | 7. Small(c) → Plant(c)                       ∀ Elim: 1
   |   | 8. Plant(c)                                  → Elim: 7,6
   |
   |   | 9. ¬Small(c)
   |   | ---
   |   | 10. ∃x ¬Small(x)                             ∃ Intro: 9
   |   | 11. ¬∃x Animal(x)                            → Elim: 10,2
   |   |   | 12. Animal(c)
   |   |   | ---
   |   |   | 13. ∃x Animal(x)                         ∃ Intro: 12
   |   |   | 14. ⊥                                    ⊥ Intro: 13,11
   |   | 15. ¬Animal(c)                               ¬ Intro: 12-14
   |   |   | 16. ¬Plant(c)
   |   |   | ---
   |   |   | 17. ¬Animal(c) ∧ ¬Plant(c)               ∧ Intro: 15,16
   |   |   | 18. (¬Animal(c) ∧ ¬Plant(c)) → Small(c)  ∀ Elim: 3
   |   |   | 19. Small(c)                             → Elim: 18,17
   |   |   | 20. ⊥                                    ⊥ Intro: 9,19
   |   | 21. Plant(c)                                 ¬ Intro: 16-20
   | 22. Plant(c)                                     ∨ Elim: 9-21,6-8,5
23. ∀x Plant(x)                                       ∀ Intro: 4-22
```

Here is a proof for Problem 5 that makes heavy use of Taut Con (especially with contrapositives), but does not use the Law of Excluded Middle.

1.	∀y(Small(y) → Plant(y))	
2.	∃x ¬Small(x) → ¬∃x Animal(x)	
3.	∀x ∀y [(¬Animal(x) ∧ ¬Plant(x)) → Small(y)]	
4.	[c]	
5.	¬Plant(c)	
6.	(¬Animal(c) ∧ ¬Plant(c)) → Small(c)	∀ Elim: 3
7.	Small(c) → Plant(c)	∀ Elim: 1
8.	¬Small(c)	Taut Con: 5,7
9.	∃x ¬Small(x)	∃ Intro: 8
10.	¬∃x Animal(x)	→ Elim: 9,2
11.	Animal(c) ∨ Plant(c)	Taut Con: 8,6
12.	Animal(c)	Taut Con: 11,5
13.	∃x Animal(x)	∃ Intro: 12
14.	⊥	⊥ Intro: 10,13
15.	Plant(c)	¬ Intro: 5-14
16.	∀x Plant(x)	∀ Intro: 4-15