Handout #33 | CS103A
October 24, 2008 | Robert Plummer

Problem Set #7

As we have mentioned in class, it takes practice to develop the skill of doing mathematical proofs and this problem set is meant to provide that practice. Because of the nature of the problems and an occasional need for "insight", it is important that you start this problem set early and do a little bit each day.

Your proofs should be semi-formal or informal, as described in Handout #29. You may always refer to a proof you have done on a previous problem set or a proof in a handout, but always include the reference. You may do statement-reason (near Fitch-style) proofs or provide concise, well-argued paragraphs. If you have any doubts as to whether your proof is formal enough, come and see us during office hours or contact us on the helpline. The proofs in the Number Theory handout are good models for your proofs on this problem set.

Note: Despite the differences in total points, all problem sets will be weighted equally when we calculate your final course grade.

Due: 11/3 (note that this is one class period later than the due date given in the Syllabus)

Warm-Up Exercises: 1 point each

Prove or disprove the following:

1) If n is an integer and $n^3 + 5$ is odd, then n is even.

2) The product of two irrational numbers is irrational.

3) If n is an integer ≥ 2, then there are no primes between n! + 2 and n! + n. (n! means n factorial, which is the product of the integers from 1 to n.)

4) If n is an odd integer, then $n^2 \equiv 1 \pmod 8$.

The Real Exercises: 3 points each

5) Prove or disprove*:* If a, b, and m are positive integers, then a mod m + b mod m = (a + b) mod m.

6) The value of the Euler ϕ function at the positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n. For example, ϕ(4) = 2, since 1 and 3 are relatively prime to 4.

Find:

a) ϕ(5)
b) ϕ(9)
c) ϕ(11)
d) ϕ(12)

Note: more problems on the back of the page!

Prove or disprove the following:

7) n is prime if and only if $\phi(n) = n-1$.

8) If a, b, and c are integers, then c | a and c | b if and only if c | gcd(a,b).

9) If a and b are integers with greatest common divisor d, then c = ax + by for some integers x and y if and only if c is a multiple of d.

10) If n is a positive integer such that the sum of its divisors is n + 1, then n is prime.

11) If ax mod n = ay mod n and gcd(a, n) = 1, then x mod n = y mod n. (a, x, n, y are positive integers)

12) If a, b, and m are integers such that $m \geq 2$ and $a \equiv b \pmod{m}$, then gcd(a,m) = gcd(b,m).

13) Suppose n has the prime-power factorization

$$n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$$

where $p_1, \quad p_2, \ \ldots \ p_k$ are distinct primes.

Show that for any integers a and b, if $a \equiv b \pmod{n}$ then

$a \equiv b \pmod{p_i^{e_i}}$ for each i = 1, ..., k.

14) Given a positive integer n, there are n consecutive odd positive integers that are primes.

Note: you will see in Handout #32 that the greatest common divisor of two integers a and b is defined only if they are not both 0. So if a problem says that d = gcd(a,b), then you may assume that at least one of a and b is not 0. Similarly, if we say a | b, then $a \neq 0$.