Handout #30 CS103A
October 22, 2008 Robert Plummer

Review Session Solutions

Possibility and Impossibility

1. For each of the following first-order logic sentences, circle ALL descriptions of the sentence which are accurate. Please interpret all predicates to have precisely the logical meaning they have in Tarski's World.

1) $\forall x(\text{Large}(x) \wedge \text{Dodec}(x)) \wedge [\neg\exists x\text{Large}(x) \vee \neg\exists x\text{Dodec}(x)]$

Tautology	First-Order Validity	Logical Truth
Tautological Possibility	First-Order Possibility	Logical Possibility
Tautological Impossibility	**First-Order Impossibility**	**Logical Impossibility**

2) $[\forall x(\text{Cube}(x) \rightarrow x=c)] \rightarrow \text{Cube}(c)$

Tautology	First-Order Validity	Logical Truth
Tautological Possibility	**First-Order Possibility**	**Logical Possibility**
Tautological Impossibility	First-Order Impossibility	Logical Impossibility

This sentence is NOT a First-Order Validity or Logical Truth, as many students claimed. A world with no cubes will falsify the statement, the statement $\forall x(\text{Cube}(x) \rightarrow x=c)$ will be vacuously true, but c will not be a cube.

3) $\exists x\text{Tet}(x) \vee \forall x\neg\text{Tet}(x)$

Tautology	**First-Order Validity**	**Logical Truth**
Tautological Possibility	**First-Order Possibility**	**Logical Possibility**
Tautological Impossibility	First-Order Impossibility	Logical Impossibility

4) $\forall x \forall y(\text{SameShape}(x, y) \leftrightarrow \text{FrontOf}(x, y))$

Tautology	First-Order Validity	Logical Truth
Tautological Possibility	**First-Order Possibility**	Logical Possibility
Tautological Impossibility	First-Order Impossibility	**Logical Impossibility**

This sentence is logically impossible because the inner statement SameShape(x, y) ↔ FrontOf(x, y) is always false for x=y. SameShape(x, x) is true and FrontOf(x, x) is false, so the biconditional will be false.

Equivalences

2. Each of the sentences in the left column has a sentence in the right column that is a First Order Equivalence. Show these equivalences by writing the sentence numbers from the left in front of the equivalent sentences on the right.

1) $\neg\exists x\exists y(P(x) \wedge \neg Q(y))$

2) $\forall x\forall y(P(x) \vee Q(y))$

3) $\neg\exists x\neg\exists y(Q(y) \rightarrow P(x))$

4) $\neg\forall x\neg\exists y(\neg P(x) \rightarrow \neg Q(y))$

3___ $\forall x\exists y(Q(y) \rightarrow P(x))$

4___ $\exists x\exists y(Q(y) \rightarrow P(x))$

1___ $\forall x\forall y(P(x) \rightarrow Q(y))$

2___ $\neg\exists x\exists y(\neg P(x) \wedge \neg Q(y))$

Conditionals

3. Prove or disprove the following. If you prove it, write up a Fitch-style proof with <u>all</u> steps included. Number your steps and refer to those numbers in the reasons you give for each step. If the argument is not valid, describe in detail where the argument fails.

You may **not** use Taut Con in your proof except for the law of the excluded middle. Note: the fact that there is nothing above the line below indicates that there are no premises for this proof.

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| 1. |
|----
| | 2. Q→R
| |----
| | | 3. P→Q
| | |----
| | | | 4. P
| | | |----
| | | | 5. Q                                  → Elim: 4,3
| | | | 6. R                                  → Elim: 5,2
| | |
| | | 7. P→R                                  → Intro: 4-6
| |
| | 8. (P→Q) → (P→R)                          → Intro: 3-7
|
| 9. (Q→R) → ((P→Q) → (P→R))                  → Intro: 2-8
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Translations

4. Translate each of the following sentences into first-order logic. The questions describe events occurring at a Halloween party. The domain of discourse includes exactly the people who attended the party. You may only use the following predicates:

$M(x)$	x wore a mask to the party.
$S(x, y)$	x scared y at the party.
$F(x, y)$	x and y are friends [Note: F is symmetric, i.e. $F(x, y) \leftrightarrow F(y, x)$]

1) Nobody who didn't wear a mask scared a friend.

$\forall x \forall y((S(x, y) \land F(x, y)) \rightarrow M(x))$
$\neg\exists x \exists y(\neg M(x) \land F(x, y) \land S(x, y))$

2) A person who wore a mask and who had no friends scared everybody else at the party.

$\exists x(M(x) \land \neg\exists y F(x, y) \land \neg\exists z(z \neq x \land \neg S(x, z))$
$\exists x \forall y (M(x) \land \neg F(x, y) \land (y \neq x \rightarrow S(x, y))$

Common Mistake: For second translation, having ($y \neq x \land S(x, y)$. This would be cause the statement to always be false, since the same object could be chosen for x and y and will be equal to itself.

3) No two people wearing masks scared each other.

$\forall x \forall y((M(x) \land M(y) \land y \neq x) \rightarrow \neg (S(x, y) \land S(y, x))$
$\neg\exists x \exists y(M(x) \land M(y) \land y \neq x \land S(x, y) \land S(x, y))$

Common Mistake: For the first translation, writing ($\neg S(x, y) \land \neg S(y, x)$). The original sentence allows that a person x wearing a mask scared another person y in a mask, as long as y did not also scare x. But this translation does not allow for that possibility. A similar mistake was having only $S(x, y)$ and not $S(y, x)$, in the second translation, for the same reason.

4) Everybody with at least two friends scared somebody.

$\forall x([\exists y \exists z(y \neq z \land F(y, x) \land F(z, x))] \rightarrow \exists w\, S(x, w))$

Common Mistake: Mis-parenthesizing by having the first three quantifiers all placed in front, ie $\forall x \exists y \exists z[(y \neq z \land F(y, x) \land F(z, x)) \rightarrow \exists w S(x, w)]$. Note this is satisfied as long as all people x are not friends with at least two people, this is the general "mixing conditional with existential quantifier" error. Also, attempting to put all four quantifiers in front is not possible, for similar reasons.

Formal Proofs with Quantifiers

5. Give a Fitch-style proof for the following. Number your steps and refer to those numbers in the reasons you give for each step. You **are allowed** to use Taut Con to justify proof steps involving only Boolean connectives.

1. $\forall x\, (\neg(D(x) \rightarrow \neg A(x)))$	
2. $\forall x\, (A(x) \rightarrow \exists y\, B(y))$	
3. $\exists x\, A(x)$	
4. [a] $A(a)$	
5. $A(a) \rightarrow \exists y\, B(y)$	∀ Elim: 2
6. $\exists y\, B(y)$	→ Elim: 4,5
7. [b] $B(b)$	
8. $\neg(D(b) \rightarrow \neg A(b))$	∀ Elim: 1
9. $D(b)$	Taut Con: 8
10. $B(b) \wedge D(b)$	Taut Con: 7,9
11. $\exists x\, (B(x) \wedge D(x))$	∃ Intro: 10
12. $\exists x\, (B(x) \wedge D(x))$	∃ Elim: 6,7-11
13. $\exists x\, (B(x) \wedge D(x))$	∃ Elim: 4-12,3