Resolution

$$\frac{P \vee Q \quad \neg Q \vee R}{P \vee R}$$

Resolution

$$\frac{P \vee Q \quad \neg Q \vee R}{P \vee R}$$

Resolution is a valid rule of inference, which we can use as a proof procedure.

To see how we do this, we have to remind ourselves about conjunctive normal form.

Conjunctive Normal Form

A sentence is in conjunctive normal form if it is a conjunction of one or more sentences, each of which is a disjunction of one or more literals.

A literal is an atomic sentence or the negation of an atomic sentence.

$(\neg A \vee C) \wedge (\neg B \vee C)$

Conjunctive Normal Form

We can put a sentence in CNF by:

- Replacing $\rightarrow$ with $\vee$
  $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

Conjunctive Normal Form

We can put a sentence in CNF by:

- Replacing $\rightarrow$ with $\vee$
  $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
- Using De Morgan's laws to move negations inward until they all apply to atomic sentences
  $\neg (P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

Conjunctive Normal Form

We can put a sentence in CNF by:

- Replacing $\rightarrow$ with $\vee$
  $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
- Using De Morgan's laws to move negations inward until they all apply to atomic sentences
  $\neg (P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
- Distribute $\vee$ over $\wedge$ to move disjunctions inward
  $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Conjunctive Normal Form

We can put a sentence in CNF by:

- Replacing $\rightarrow$ with $\vee$
  $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
- Using De Morgan's laws to move negations inward until they all apply to atomic sentences
  $\neg (P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
- Distribute $\vee$ over $\wedge$ to move disjunctions inward
  $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
- Use the distributive law to flatten nested disjunctions and conjunctions
  $P \vee (Q \vee R) \Leftrightarrow (P \vee Q \vee R)$

Conjunctive Normal Form

We can put a sentence in CNF by:

- Replacing $\rightarrow$ with $\vee$
  $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
- Using De Morgan's laws to move negations inward until they all apply to atomic sentences
  $\neg (P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
- Distribute $\vee$ over $\wedge$ to move disjunctions inward
  $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
- Use the distributive law to flatten nested disjunctions and conjunctions
  $P \vee (Q \vee R) \Leftrightarrow (P \vee Q \vee R)$
- Eliminate double negatives
  $\neg\neg P \Leftrightarrow P$

Conjunctive Normal Form

$$\neg((A \vee B) \wedge \neg C) \Leftrightarrow \neg(A \vee B) \vee \neg\neg C$$
$$\Leftrightarrow \neg(A \vee B) \vee C$$
$$\Leftrightarrow (\neg A \wedge \neg B) \vee C$$
$$\Leftrightarrow (\neg A \vee C) \wedge (\neg B \vee C)$$

Resolution

When doing resolution, it is conventional to write the sentence we are examining as a set of clauses, each of which is a set of literals.

CNF:

$\neg A \wedge (B \vee C \vee B) \wedge (\neg C \vee \neg D) \wedge (A \vee D) \wedge (\neg B \vee \neg D)$

Clauses:

$\{\neg A\}, \{B, C\}, \{\neg C, \neg D\}, \{A, D\}, \{\neg B, \neg D\}$

Resolution

A clause R is a resolvent clauses C1 and C2 if there is an atomic sentence in one of the clauses whose negation is in the other clause and R is the set of all the other literals in either clause.

$$\frac{\{A, D\}\ \{\neg A\}}{\{D\}} \qquad \frac{\{B, C\}\ \{\neg B, \neg D\}}{\{C, \neg D\}} \qquad \frac{\{D\}\ \{\neg D\}}{\square}$$

Consequence

S is a tautological consequence of premises $P_1, \ldots P_n$

if and only if the set $\{P_1, \ldots P_n, \neg S\}$ is not satisfiable,

that is, if $P_1 \wedge \ldots \wedge P_n \wedge \neg S$ is not tt-satisfiable.

Consequence

S is a tautological consequence of premises $P_1, \ldots P_n$ if and only if the set $\{P_1, \ldots P_n, \neg S\}$ is not satisfiable,

that is, if $P_1 \wedge \ldots \wedge P_n \wedge \neg S$ is not tt-satisfiable.

If $P_1 \wedge \ldots \wedge P_n \wedge \neg S$ is not tt-satisfiable, we can use resolution to derive a contradiction from it.

Soundness and Completeness

A deductive system is sound if everything that we can prove from a set of premises is indeed a first-order consequence of the premises.

A deductive system is complete if everything that is a first-order consequence of a set of premises can actually be proved.

Soundness Theorem for Resolution: If S is a set of clauses, and it is possible to arrive at □ by successive resolutions, then S is not tt-satisfiable.

Completeness Theorem for Resolution: If S is a set of clauses that is not tt-satisfiable, then it is always possible to arrive at □ by successive resolutions.

Soundness Theorem for Resolution: If S is a set of clauses, and it is possible to arrive at □ by successive resolutions, then S is not tt-satisfiable.

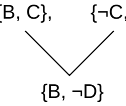
Completeness Theorem for Resolution: If S is a set of clauses that is not tt-satisfiable, then it is always possible to arrive at □ by successive resolutions.

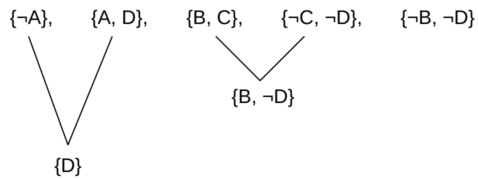
If we want to show $P_1, \ldots P_n \vdash S$, we can use resolution to derive □ from the clauses of $P_1, \ldots P_n, \neg S$.

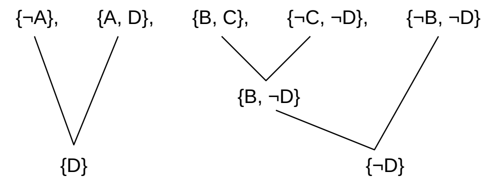
($P_1, \ldots P_n \vdash S$ means that there is a formal proof of S starting with the P's as premises.)

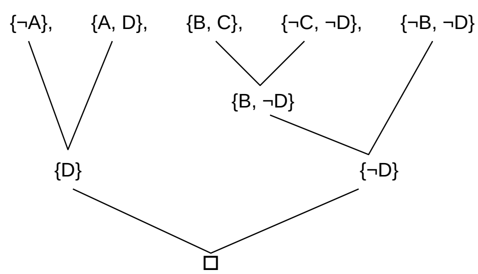
{¬A}, {A, D}, {B, C}, {¬C, ¬D}, {¬B, ¬D}

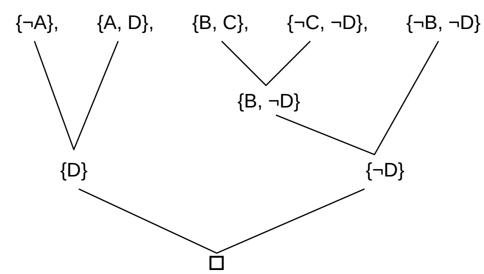
{¬A}, {A, D}, {B, C}, {¬C, ¬D}, {¬B, ¬D}











This is a fairly mechanical process that can be automated.

We can extend resolution to sentences with quantifiers.

Sally is studying with Frederico.

Frederico is at Tresidder.

If any person is studying with another person who is at a particular place, the first person is also at that place.

If someone is at a particular place, then he or she can be reached at the phone number for that place.

There is a phone number where Sally can be reached.

Predicates:

- SW(x, y): x is studying with y
- A(x,y): x is at place y
- R(x,y): x can be reached at phone number y
- ph(x): phone number of place x

Here are the premises:

- SW(Sally, Frederico)
- A(Frederico, Tresidder)
- $\forall x\ \forall y\ \forall z\ (SW(x,y) \wedge A(y,z) \rightarrow A(x,z))$
- $\forall x\ \forall y\ (A(x, y) \rightarrow R(x, ph(y)))$

Negation of the conclusion:

$\neg\exists x R(Sally, ph(x))$

C1: SW(Sally, Frederico)
C2: A(Frederico, Tresidder)
C3: ¬SW(x,y) v ¬A(y,z) v A(x,z)
C4: ¬A(u, v) v R(u, ph(v))
C5: ¬R(Sally, ph(w))

The resolution steps that produce the null clause are:

C6: ¬A(Sally, v)	resolution and unification of C4 and C5 Sally/u, v/w

---

C1: SW(Sally, Frederico)
C2: A(Frederico, Tresidder)
C3: ¬SW(x,y) v ¬A(y,z) v A(x,z)
C4: ¬A(u, v) v R(u, ph(v))
C5: ¬R(Sally, ph(w))

The resolution steps that produce the null clause are:

C6: ¬A(Sally, v)	resolution and unification of C4 and C5 Sally/u, v/w
C7: ¬SW(Sally, y) v ¬A(y, v)	resolution and unification of C3 and C6 Sally/x, v/z

---

C1: SW(Sally, Frederico)
C2: A(Frederico, Tresidder)
C3: ¬SW(x,y) v ¬A(y,z) v A(x,z)
C4: ¬A(u, v) v R(u, ph(v))
C5: ¬R(Sally, ph(w))

The resolution steps that produce the null clause are:

C6: ¬A(Sally, v)	resolution and unification of C4 and C5 Sally/u, v/w
C7: ¬SW(Sally, y) v ¬A(y, v)	resolution and unification of C3 and C6 Sally/x, v/z
C8: ¬SW(Sally, Frederico)	resolution and unification of C2 and C7 Frederico/y, Tresidder/v

---

C1: SW(Sally, Frederico)
C2: A(Frederico, Tresidder)
C3: ¬SW(x,y) v ¬A(y,z) v A(x,z)
C4: ¬A(u, v) v R(u, ph(v))
C5: ¬R(Sally, ph(w))

C6: ¬A(Sally, v)	resolution and unification of C4 and C5 Sally/u, v/w
C7: ¬SW(Sally, y) v ¬A(y, v)	resolution and unification of C3 and C6 Sally/x, v/z
C8: ¬SW(Sally, Frederico)	resolution and unification of C2 and C7 Frederico/y, Tresidder/v
C9: □	resolution of C1 and C8

C1: SW(Sally, Frederico)
C2: A(Frederico, Tresidder)
C3: ¬SW(x,y) v ¬A(y,z) v A(x,z)
C4: ¬A(u, v) v R(u, ph(v))
C5: ¬R(Sally, ph(w)) v Ans(w)

C1: SW(Sally, Frederico)
C2: A(Frederico, Tresidder)
C3: ¬SW(x,y) v ¬A(y,z) v A(x,z)
C4: ¬A(u, v) v R(u, ph(v))
C5: ¬R(Sally, ph(w)) v Ans(w)

C6: ¬A(Sally, v) v Ans(v)	C4 and C5 Sally/u, v/w
C7: ¬SW(Sally, y) v ¬A(y, v) v Ans(v)	C3 and C6 Sally/x, v/z
C8: ¬SW(Sally, Frederico) v Ans(Tresidder)	C2 and C7 Frederico/y, Tresidder/v
C9: Ans(Tresidder)	C1 and C8

Universal 2

$\forall x\,(Tet(x) \rightarrow \forall y\,(Cube(y) \rightarrow LeftOf(x, y)))$

$\forall x\,\forall y\,((Tet(x) \wedge Cube(y)) \rightarrow LeftOf(x, y))$

Universal 2

$\forall x\,\forall y\,(Tet(x) \rightarrow (Cube(y) \rightarrow LeftOf(x, y)))$

$\forall x\,\forall y\,((Tet(x) \wedge Cube(y)) \rightarrow LeftOf(x, y))$

Universal 2

$\forall x\,\forall y\,(Tx \rightarrow (Cy \rightarrow Lxy))$

$\forall x\,\forall y\,((Tx \wedge Cy)) \rightarrow Lxy)$

Universal 2

$Tx \rightarrow (Cy \rightarrow Lxy)$ $\qquad \neg((Tx \wedge Cy) \rightarrow Lxy)$

Universal 2

$Tx \rightarrow (Cy \rightarrow Lxy)$ $\qquad \neg((Tx \wedge Cy) \rightarrow Lxy)$

$\neg Tx \vee \neg Cy \vee Lxy$

Universal 2

$Tx \rightarrow (Cy \rightarrow Lxy)$ $\qquad \neg((Tx \wedge Cy) \rightarrow Lxy)$

$\neg Tx \vee \neg Cy \vee Lxy$

$\{\neg Tx, \neg Cy, Lxy\}$

Universal 2

$Tx \rightarrow (Cy \rightarrow Lxy)$ $\neg((Tx \wedge Cy) \rightarrow Lxy)$
$\neg Tx \vee \neg Cy \vee Lxy$ $\neg(\neg(Tx \wedge Cy) \vee Lxy)$

$\{\neg Tx, \neg Cy, Lxy\}$

Universal 2

$Tx \rightarrow (Cy \rightarrow Lxy)$ $\neg((Tx \wedge Cy) \rightarrow Lxy)$
$\neg Tx \vee \neg Cy \vee Lxy$ $\neg(\neg(Tx \wedge Cy) \vee Lxy)$
$Tx \wedge Cy \wedge \neg Lxy$

$\{\neg Tx, \neg Cy, Lxy\}$

Universal 2

$Tx \rightarrow (Cy \rightarrow Lxy)$ $\neg((Tx \wedge Cy) \rightarrow Lxy)$
$\neg Tx \vee \neg Cy \vee Lxy$ $\neg(\neg(Tx \wedge Cy) \vee Lxy)$
$Tx \wedge Cy \wedge \neg Lxy$

$\{\neg Tx, \neg Cy, Lxy\}$ $\{Tx\}$ $\{Cy\}$ $\{\neg Lxy\}$

Resolution theorem proving has seen many applications in the field of Artificial Intelligence (AI), such as robot problem solving and natural language processing.

Resolution is the basis of the computer language Prolog, and a related technique is used in Fitch.

Finding a proof can involve a large amount of search, and many strategies have been devised to improve the efficiency of the process.

References

J.A. Robinson, “A machine-oriented logic based on the resolution principle,” *Journal of the Association for Computing Machinery*, 12(1), 1965.

Nilsson, Nils J., *Artificial Intelligence: A New Synthesis*, San Francisco: Morgan Kaufmann, 1998.

Midterm

- Taut/FO/Logical consequence, etc.
- Equivalences
- Proof with conditionals
- Translations
- Proof with quantifiers

Overall Strategy

The problems are (roughly) in order of increasing difficulty. Do them in the order that works best for you.

Don't spend too much time on any one problem. If you get stuck, do the other problems and come back.

Check your work.

Proof Strategy

Begin by deciding why the assertion is true (if it is).

An informal proof can be a good starting point.

Use the rules of inference, which will be on the last two pages of the exam.

For rules involving subproofs, lay out the structure first.

Try working backwards from the conclusion.

Don't forget proof by contradiction and the law of the excluded middle.

Write legibly!

Do not give more than one answer to a question.

Good Luck!