Introduction to Quantification

Predicates Revisited

We have been working with predicates since the beginning of our studies in logic. Predicate symbols are used to express some property of objects or some relation between objects. Our discussion of predicates has been limited to those that refer to specific objects, e.g., Boy(Elliot) or Tet(b) where b is an actual entity in Tarski's World. We will now expand this notion to allow **variables** to be used with predicates. So, we can now refer to Boy(x) or Apple(y) where the value of x and y may not be known.

In general, we think of logic using predicates and variables (i.e., predicate logic) as a more "powerful" language. We can manipulate the variables within a predicate in a way that is just not possible with propositional logic. In fact, predicate logic is expressive enough to form the basis of a number of useful programming languages including Prolog, and SQL (structured-query language). Predicate logic is also used in reasoning systems or "expert" systems, such as medical-diagnosis programs and theorem-proving programs.

When we work with predicate logic, the variables take on an important role. It is by defining a value for these variables that we can actually assign a true or false value to the predicate. For example, Apple(y) is true if and only if y is an apple. If y happens to a plum, then Apple(y) is false.

The collection of values that can replace a variable in a predicate is called the **universe** or **domain of discourse** of the predicate. In an arity-one predicate, if a value from the universe can be substituted for the variable to make it true, we say the value **satisfies** the predicate, and the predicate is said to be **satisfiable**. The same ideas are extended to arity-n predicates.

By itself, we can't determine the truth value of a formula like Cube(x). But we can determine the value if we say what objects the formula applies to, in sentences like

For some x in the world, Cube(x).
For every x in the world, Cube(x).

The first sentence tells us of the existence of a cube, and the second says that being a cube is a universal property of objects in the world we are discussing. We have a notation for writing these statements:

$\exists x Cube(x)$
$\forall x Cube(x)$

The symbol "∃" is called the **existential quantifier**, and "∀" is called the **universal quantifier**. If all the variables in a predicate have been quantified, we can determine if it is true.

Example

What is the truth value of $\forall x\, P(x)$ where $P(x)$ is the statement $x + 1 > x$ and the universe of the predicate is all real numbers?

Since $P(x)$ is true for all real numbers, the quantification is true.

Example

What is the truth value of $\exists x\, P(x)$ where $P(x)$ is the statement $x = x + 1$ and the universe of the predicate is all real numbers?

Since $P(x)$ is false for all real numbers, the quantification is false.

Summary of Quantifiers in Arity-One Predicates

Statement	When true?	When false?
$\forall x\, P(x)$	P(x) is true for all x	an x exists where P(x) is false
$\exists x\, P(x)$	There is an x where P(x) is true	P(x) is false for all x

A variable that has been quantified is called a **bound** variable. Variables that have not been bound are **free**. With the "binding" of a variable, it becomes possible to determine the truth value of the predicate.

Typically, we use quantifiers to make conditional claims like the following:

$\forall x\, (\text{Professor}(x) \rightarrow \text{Smart}(x))$
$\exists x\, (\text{Professor}(x) \wedge \text{Smart}(x))$

The first translates to: Every professor is smart. Or, to be more exact, if you pick anything at all, you’ll find it is either not a professor or that it is smart (or maybe both). The second has a different meaning: Some professor is smart, or, to put it another way, there is at least one smart professor.

Some students new to predicate logic wonder why “Every professor is smart” cannot be represented as

$\forall x\, (\text{Professor}(x) \wedge \text{Smart}(x))$

But this does not have the same meaning as “Every professor is smart”. Do you know why?

One of the most powerful features of predicate logic is its ability to capture real-world facts in a precise, compact way. This is one of the reasons why it has been used in artificial intelligence as a knowledge representation scheme. We will practice a great deal with translation from English sentences into quantified predicates, variables and Boolean connectives.

Well-Formed Formulas

Apple(y) or Boy(x) are examples of **atomic well-formed formulas (wff's)**. An atomic wff is any n-ary predicate followed by n terms where the terms can be variables or constants.

The following examples are wff's, but they are not atomic:

$\forall x\ (\text{Professor}(x) \rightarrow \text{Smart}(x))$
$\exists x\ (\text{Professor}(x) \wedge \text{Smart}(x))$

A wff is formed as follows:

0. Atomic wffs as defined above are, of course, wffs.
1. If P is a wff, then so is ~P.
2. If $P_1, \ldots, P_n$ are wffs, so is $(P_1 \wedge \ldots \wedge P_n)$.
3. If $P_1, \ldots, P_n$ are wffs, so is $(P_1 \vee \ldots \vee P_n)$.
4. If P and Q are wffs, so is $(P \rightarrow Q)$.
5. If P and Q are wffs, so is $(P \leftrightarrow Q)$.
6. If P is a wff, and x is a variable, then $\forall x$ P is a wff and any occurrence of x in $\forall x$ P is said to be bound.
7. If P is a wff, and x is a variable, then $\exists x$ P is a wff and any occurrence of v in $\exists x$ P is said to be bound.

A **sentence** is a wff with no free variables. To determine if a wff is a sentence we often have to consider the scope of a quantifier. Parentheses are essential in helping to define this. For example, the first wff below is a sentence since x is bound, but the second is not since the x in Smart(x) is free.

$\exists x\ (\text{Professor}(x) \wedge \text{Smart}(x))$
$\exists x\ \ \text{Professor}(x) \wedge \text{Smart}(x)$

If replacing all occurrences of the free variable x in a wff S(x) with the object b produces a true statement, then we say that S is satisfied by b.

A sentence of the form $\forall x$ S(x) is true if and only if the wff S(x) is satisfied by every object in the domain of discourse.

A sentence of the form $\exists x$ S(x) is true if and only if the wff S(x) is satisfied by at least one object in the domain of discourse.

The Four Aristotelian Forms:

- All P's are Q's — $\forall x\ (P(x) \rightarrow Q(x))$
- Some P's are Q's — $\exists x\ (P(x) \wedge Q(x))$
- No P's are Q's — $\forall x\ (P(x) \rightarrow \neg Q(x))$
- Some P's are not Q's — $\exists x\ (P(x) \wedge \neg Q(x))$

Translations

As we mentioned earlier, one of the most powerful aspects of predicate logic and quantifiers is how they allow us to capture real-world facts in a concise form. But it takes practice in learning how to do these translations.

Lewis Carroll (the author of Alice in Wonderland) wrote several books on symbolic logic. He gives many examples of reasoning using quantifiers:

Example

All lions are fierce.
Some lions do not drink coffee.
Some fierce creatures do not drink coffee.

If P(x) denotes "x is a lion"; Q(x) denotes "x is fierce"; and R(x) denote "x drinks coffee", we can express the 3 statements above using quantifiers and P(x), Q(x), and R(x) (assuming the universe is the set of all creatures).

$\forall x\ (P(x) \rightarrow Q(x))$
$\exists x\ (P(x) \wedge \neg R(x))$
$\exists x\ (Q(x) \wedge \neg R(x))$

Problem

P(x): “x is a professor”
I(x): “x is ignorant”
V(x): “x is vain”

Express the following statements using quantifiers, logical connectives and P(x), I(x), and V(x), where the universe is the set of all people.

a) No professors are ignorant.

b) All ignorant people are vain.

c) No professors are vain.

Historical Notes

The formal concept of a quantifier was introduced into the symbolic logic of George Boole by the American philosopher, logician and engineer **Charles Sanders Peirce** (1839 - 1914), who pronounced his name "purse". He was the founder of pragmatism, a branch of philosophy which attempts to bring philosophical and intellectual pursuits into the realm of the practical. Peirce made his living working for the US Coast Survey doing gravity research, and he was the first person to propose using the wavelength of light as a unit of measurement. His primary interests were in logic and philosophy, and he made significant contributions to chemistry, physics, astronomy, geodesy, meteorology, engineering, cartography, psychology, philology, the history and philosophy of science and mathematics, and phenomenology. He was a very gifted scientist and logician; he was also ambidextrous such that he could write a question with one hand, and write the answer simultaneously with the other.

(Original handout by Maggie Johnson)