CS103A

10/10/08

Midterm Exam

Thurs., Oct. 23
7 - 9 pm
Location TBA

8.38.2

$\neg(P \wedge Q) \vee P$

8.38.2

$P \vee \neg P$ — Taut Con

$\neg(P \wedge Q) \vee P$ — ∨ Elim

8.38.2

$P \vee \neg P$ — Taut Con

P

$\neg(P \wedge Q) \vee P$ — ∨ Intro

$\neg P$

$\neg(P \wedge Q) \vee P$

$\neg(P \wedge Q) \vee P$ — ∨ Elim

8.38.2

$P \vee \neg P$ — Taut Con

P

$\neg(P \wedge Q) \vee P$ — ∨ Intro

$\neg P$

$\neg(P \wedge Q)$

$\neg(P \wedge Q) \vee P$ — ∨ Intro

$\neg(P \wedge Q) \vee P$ — ∨ Elim

8.38.2

$P \vee \neg P$ — Taut Con

P

$\neg(P \wedge Q) \vee P$ — ∨ Intro

$\neg P$

$P \wedge Q$

P — ∧ Elim

$\bot$ — ⊥ Intro

$\neg(P \wedge Q)$ — ¬ Intro

$\neg(P \wedge Q) \vee P$ — ∨ Intro

$\neg(P \wedge Q) \vee P$ — ∨ Elim

8.29

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|
|---
| | P → Q
| | P ∨ ¬P                      ✓ Taut Con
| | | P
| | | Q                         ✓ → Elim
| | | ¬P ∨ Q                    ✓ ∨ Intro
| | | ¬P
| | | ¬P ∨ Q                    ✓ ∨ Intro
| | ¬P ∨ Q                      ✓ ∨ Elim
| | ¬P ∨ Q
| | | P
| | | | ¬P
| | | | ⊥                       ✓ ⊥ Intro
| | | | Q                       ✓ ⊥ Elim
| | | | Q
| | | Q                         ✓ ∨ Elim
| | P → Q                       ✓ → Intro
| (P → Q) ↔ (¬P ∨ Q)            ✓ ↔ Intro
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8.29

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|---
| | P → Q
| |
| | ¬P ∨ Q
| | ¬P ∨ Q
| |
| | P → Q
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8.29

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| | | Q                         ✓ → Elim
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| | | ¬P
| | | ¬P ∨ Q                    ✓ ∨ Intro
| | ¬P ∨ Q                      ✓ ∨ Elim
| | ¬P ∨ Q
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8.29

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| | | | ¬P
| | | | ⊥                       ✓ ⊥ Intro
| | | | Q                       ✓ ⊥ Elim
| | | | Q
| | | Q                         ✓ ∨ Elim
| | P → Q                       ✓ → Intro
| (P → Q) ↔ (¬P ∨ Q)            ✓ ↔ Intro
```

Quantification

So far we have used

-- propositions and connectives } Propositional calculus

-- identity

-- predicates

Quantification

So far we have used

-- propositions and connectives

-- identity

-- predicates

} FOL

Quantification

-- propositions and connectives

-- identity

-- predicates

-- quantifiers

} FOL

Predicate calculus

Quantification

We are after more expressive power.

"There is a cube in the world."

"There are no tetrahedrons in the world."

"All cubes in the world are large."

Quantification

The first thing we need are variables.

We want to say that x is a variable and write

Cube(x)

That is not a sentence in FOL. Since it does not specify what x refers to, we can't say whether it is true or false.

We say that x is a "free variable" in the formula.

Quantification

We can make sentences out of formulas with variables:

"There is an object x such that Cube(x)."

"For every object x, Cube(x)."

"For every object x, ¬Cube(x)."

"It is not the case that there is an object x such that Cube(x)."

Quantification

To write these sentences in FOL, we need some notation.

$\exists x$ Cube(x) — $\exists$ is the Existential Quantifier

$\forall x$ Cube(x) — $\forall$ is the Universal Quantifier

Quantification

To write these sentences in FOL, we need some notation.

$\exists x$ Cube(x) — $\exists$ is the Existential Quantifier

For some object x, Cube(x)

$\forall x$ Cube(x) — $\forall$ is the Universal Quantifier

For every object x, Cube(x)

We say that the quantifiers are "binding operators". Once we apply them, we have a sentence that is true or false.

Quantification

When we use variables, we understand that they refer to objects in a universe of discourse.

If the universe is all real numbers,

$\forall x\ (x + 1 > 0)$ is false

If the universe is all positive numbers,

$\forall x\ (x + 1 > 0)$ is true

Quantification

We often use quantifiers with conditionals.

Every professor is smart:

$\forall x$ (Professor(x) $\rightarrow$ Smart(x))

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Why not:

$\forall x$ (Professor(x) $\wedge$ Smart(x))

Quantification

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Every professor is smart:

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Why not:

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Some professor is smart:

Quantification

We often use quantifiers with conditionals.

Every professor is smart:

$\forall x$ (Professor(x) $\rightarrow$ Smart(x))

Why not:

$\forall x$ (Professor(x) $\wedge$ Smart(x))

Some professor is smart:

$\exists x$ (Professor(x) $\wedge$ Smart(x))

Well-formed Formulas (wff's)

In a formal language, we need to say how we can create meaningful expressions.

For example, P$\wedge\wedge\wedge\wedge$ uses symbols from FOL but is not meaningful.

See the text and p. 3 of the handout for the rules for creating wff's.

A sentence is a wff with no free variables.

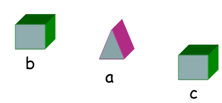
We use the notation S(x) to stand for a wff which has x as its only free variable.

If b is an object, then S(b) is the sentence we get when we replace all free occurrences of x in S with b. If S(b) is true, we say that b satisfies S(x).

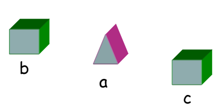
When is the sentence $\exists x$ S(x) true?

$\exists x$ S(x) is true if and only if there is at least one object that satisfies S(x).

S(x): Cube(x) $\wedge$ LeftOf(x, a)



S(x): Cube(x) $\wedge$ LeftOf(x, a)



S(b) is true
S(c) is false
$\exists x$ S(x) is true because b satisfies S
$\forall x$ S(x) is false because not every object satisfies S

By the way...

-- We assume that the domain of discourse is not empty; i.e., that it contains at least one object.

-- We assume that every constant that we use stands for an object in the domain.

	When true?	When false?
∀x P(x)	P(x) true for every x	There is an x such that P(x) is false
∃x P(x)	There is an x such that P(x) is true	P(x) is false for every x

	When true?	When false?
∀x P(x)	P(x) true for every x	There is an x such that P(x) is false ∃x ¬P(x)
∃x P(x)	There is an x such that P(x) is true	P(x) is false for every x

	When true?	When false?
∀x P(x)	P(x) true for every x	There is an x such that P(x) is false ∃x ¬P(x) ⇔ ¬∀x P(x)
∃x P(x)	There is an x such that P(x) is true	P(x) is false for every x

	When true?	When false?
∀x P(x)	P(x) true for every x	There is an x such that P(x) is false ∃x ¬P(x) ⇔ ¬∀x P(x)
∃x P(x)	There is an x such that P(x) is true	P(x) is false for every x ∀x ¬P(x)

	When true?	When false?
∀x P(x)	P(x) true for every x	There is an x such that P(x) is false ∃x ¬P(x) ⇔ ¬∀x P(x)
∃x P(x)	There is an x such that P(x) is true	P(x) is false for every x ∀x ¬P(x) ⇔ ¬∃ x P(x)

	When true?	When false?
∀x P(x)	P(x) true for every x ∀x P(x) ⇔ ¬∃x ¬P(x)	There is an x such that P(x) is false ∃x ¬P(x) ⇔ ¬∀x P(x)
∃x P(x)	There is an x such that P(x) is true	P(x) is false for every x ∀x ¬P(x) ⇔ ¬∃ x P(x)

	When true?	When false?
$\forall x\, P(x)$	P(x) true for every x $\forall x\, P(x) \Leftrightarrow \neg\exists x\, \neg P(x)$	There is an x such that P(x) is false $\exists x\, \neg P(x) \Leftrightarrow \neg\forall x\, P(x)$
$\exists x\, P(x)$	There is an x such that P(x) is true $\exists x\, P(x) \Leftrightarrow \neg\forall x\, \neg P(x)$	P(x) is false for every x $\forall x\, \neg P(x) \Leftrightarrow \neg\exists x\, P(x)$

Aristotelian Forms

All P's are Q's

Some P's are Q's

No P's are Q's

Some P's are not Q's

Aristotelian Forms

All P's are Q's
$\forall x(P(x) \rightarrow Q(x))$
All members of class P are in class Q. All men are mortal.

Some P's are Q's
$\exists x(P(x) \wedge Q(x))$
Classes P and Q have at least one object in common. Some cars are green.

No P's are Q's
$\forall x(P(x) \rightarrow \neg Q(x))$
Classes P and Q have no members in common. No strawberries are blue.

Some P's are not Q's
$\exists x(P(x) \wedge \neg Q(x))$
At least one member of class P is not a member of class Q. Some cars are not green.

Aristotelian Forms

All P's are Q's .
$\forall x(P(x) \rightarrow Q(x))$
$\neg\exists x(P(x) \wedge \neg Q(x))$

Some P's are Q's
$\exists x(P(x) \wedge Q(x))$
$\neg\forall x(P(x) \rightarrow \neg Q(x))$

No P's are Q's.
$\forall x(P(x) \rightarrow \neg Q(x))$
$\neg\exists x(P(x) \wedge Q(x))$

Some P's are not Q's
$\exists x(P(x) \wedge \neg Q(x))$
$\neg\forall x(P(x) \rightarrow Q(x))$

Aristotelian Forms

Complicated sentences can still be Aristotelian forms.

$\forall x\, [\, Cube(x) \rightarrow \exists y\, (Tet(y) \wedge LeftOf(x, y))\,]$

is an example of the first form

$\forall x\, (\, P(x) \rightarrow Q(x)\,)$

It says that every cube in the world under consideration has the property of being to the left of a tetrahedron.

Notice that implication in a sentence like

$\forall x\, (P(x) \rightarrow Q(x))$ seems quite natural.

But what does $\exists x\, (P(x) \rightarrow Q(x))$ mean?

It's better to write the second sentence as

$\exists x\, (\neg P(x) \vee Q(x))$

Translations

P(x): x is a professor
I(x): x is ignorant
V(x): x is vain

No professors are ignorant

All ignorant people are vain

No professors are vain

Translations

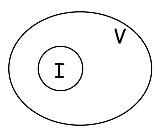
P(x): x is a professor
I(x): x is ignorant
V(x): x is vain

No professors are ignorant $\forall x(P(x) \rightarrow \neg I(x))$

All ignorant people are vain $\forall x(I(x) \rightarrow V(x))$

No professors are vain $\forall x(P(x) \rightarrow \neg V(x))$

Translations



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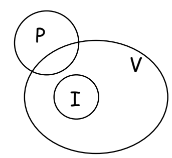

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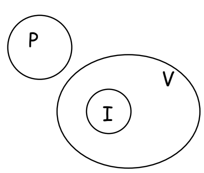

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→ All ignorant people are vain $\forall x(I(x) \rightarrow V(x))$

→ No professors are vain $\forall x(P(x) \rightarrow \neg V(x))$

Translations

S(x): x is a student in this class
C(x): x has visited Canada
M(x): x has visited Mexico

Some student in this class has visited Mexico.

$\exists x (S(x) \wedge M(x))$ or is it $\exists x (S(x) \rightarrow M(x))$

Every student in the class has visited Canada or Mexico.

$\forall x (S(x) \rightarrow [C(x) \vee M(x)])$