

Conditionals

P	Q	$P \text{ ? } Q$
T	T	T
T	F	F
F	T	T
F	F	T

Observation 1: The ? connective is pretty weak. Knowing that $P \text{ ? } Q$ is true only eliminates one possible combination of P and Q.

Observation 2: If we know $P \text{ ? } Q$ is true and we also know P is false, then we have no information about the truth of Q.

Observation 3: If we know $P \text{ ? } Q$ is true and we also know P is true, then we are sure Q is true.

Conditionals

P	Q	$P \text{ ? } Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
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Conditionals

P	Q	$P \rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
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\rightarrow is called the **material conditional**, meaning that it is **truth-functional**

Conditionals

P	Q	$P \rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

The best rendering of $P \rightarrow Q$ in English is "if P, then Q"

Antecedent
Consequent

P	Q	$P \rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

If the department gets an extra \$40K, we'll buy a new copier.

Is this true:

\$40K	New Copier	$\$40K \rightarrow \text{New Copier}$

P	Q	$P \rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

If the moon is not made of cheese, then triangles have three sides.

If the moon is made of cheese, then triangles have three sides.

If the moon is made of cheese, then triangles are round.

If the moon is made of cheese, then the moon is not made of cheese.

P only if Q is translated as $P \rightarrow Q$

If P is true, then Q must be true, because P is true only if Q is true.

It will snow only if it is cold.

P only if Q is translated as $P \rightarrow Q$

If P is true, then Q must be true, because P is true only if Q is true.

It will snow only if it is cold.

Cold is a necessary condition for snow.

Snow \rightarrow Cold

If it's not cold, it will not snow.

\neg Cold \rightarrow \neg Snow

P only if Q is translated as $P \rightarrow Q$

P if Q is translated as $Q \rightarrow P$

P if and only if Q iff is translated as $(Q \rightarrow P) \wedge (P \rightarrow Q)$

P only if Q is translated as $P \rightarrow Q$

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P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

P only if Q is translated as $P \rightarrow Q$

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P if and only if Q iff is translated as $(Q \rightarrow P) \wedge (P \rightarrow Q)$

P	Q	$P \leftrightarrow Q$	$(Q \rightarrow P) \wedge (P \rightarrow Q)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Biconditional

Biconditional

$P \leftrightarrow Q$ is equivalent to $(Q \rightarrow P) \wedge (P \rightarrow Q)$

$P \leftrightarrow Q \iff (Q \rightarrow P) \wedge (P \rightarrow Q)$

Useful Equivalences involving Conditionals

$$P \rightarrow Q \Leftrightarrow \neg P \vee Q$$

$$P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$$

$$P \rightarrow Q \Leftrightarrow \neg(P \wedge \neg Q)$$

$$P \vee Q \Leftrightarrow \neg P \rightarrow Q$$

$$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$$

$$P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

Rules of Inference

\rightarrow Elim

$P \rightarrow Q$ P	Q
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Modus ponens

Rules of Inference

\rightarrow Elim

$P \rightarrow Q$ P	Q
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Modus ponens

We can prove Modus tollens

$P \rightarrow Q$ $\neg Q$	<table style="border-collapse: collapse;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px; vertical-align: top;"> P </td> <td style="padding-left: 5px; vertical-align: top;"> Q </td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px; vertical-align: top;"> \perp </td> <td style="padding-left: 5px; vertical-align: top;"> \rightarrow Elim \perp Intro </td> </tr> </table>	P	Q	\perp	\rightarrow Elim \perp Intro
P	Q				
\perp	\rightarrow Elim \perp Intro				

\neg Intro

Rules of Inference

\rightarrow Elim

$P \rightarrow Q$ P	Q
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Modus ponens

\rightarrow Intro

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P	Q		

Rules of Inference

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$P \rightarrow Q$ P	Q
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Modus ponens

\rightarrow Intro

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P	Q						
P	$(P \wedge R) \rightarrow Q$						
$R \rightarrow Q$	$R \rightarrow Q$						

Rules of Inference

\rightarrow Elim

$P \rightarrow Q$ P	Q
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Modus ponens

\rightarrow Intro

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P	Q													
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R	$P \wedge R$													
Q	Q													
$R \rightarrow Q$	$R \rightarrow Q$	\rightarrow Intro												

Rules of Inference

<p>→ Elim</p> $\frac{P \rightarrow Q \quad P}{Q}$	<p>↔ Elim</p> $\frac{P \leftrightarrow Q \text{ (or } Q \leftrightarrow P)}{P}$
<p>→ Intro</p> $\frac{\begin{array}{ l} P \\ \hline Q \end{array}}{P \rightarrow Q}$	<p>↔ Intro</p> $\frac{\begin{array}{ l} P \\ \hline Q \\ \hline P \end{array}}{P \leftrightarrow Q}$

If we have this proof:

$$\frac{}{P} \quad \frac{}{Q}$$

We could also do this:

$$\frac{\begin{array}{|l} P \\ \hline Q \end{array}}{P \rightarrow Q}$$

and vice versa

So Q is a logical consequence of P iff $P \rightarrow Q$ is a logical truth

8.48

- Small(a) ∧ (Medium(b) ∨ Large(c))
- Medium(b) → FrontOf(a, b)
- Large(c) → Tet(c)

- ¬Tet(c) → FrontOf(a, b)

8.48

- Small(a) ∧ (Medium(b) ∨ Large(c))
- Medium(b) → FrontOf(a, b)
- Large(c) → Tet(c)
- Medium(b) ∨ Large(c) ✓ ∨ ∧ Elim
- ¬Tet(c)

- FrontOf(a, b)
- ¬Tet(c) → FrontOf(a, b) ✓ ∨ → Intro

8.48

- Small(a) ∧ (Medium(b) ∨ Large(c))
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- ¬Tet(c)
- Medium(b) ✓ ∨ → Elim
- FrontOf(a, b) ✓ ∨ → Elim
- Large(c) ✓ ∨ → Elim
- Tet(c) ✓ ∨ → Elim
- ⊥ ✓ ∨ ⊥ Intro
- FrontOf(a, b) ✓ ∨ → Elim
- FrontOf(a, b)
- ¬Tet(c) → FrontOf(a, b) ✓ ∨ → Intro

8.35

- ¬Cube(b) → Small(b)
- Small(c) → (Small(d) ∨ Small(e))
- Small(d) → ¬Small(c)
- Cube(b) → ¬Small(e)

- Small(c) → Small(b)

8.38.1

$$\begin{array}{|l} \text{a} \\ \hline (P \wedge Q) \rightarrow P \end{array}$$

8.38.1

$$\begin{array}{|l} \text{a} \\ \hline \text{v } P \wedge Q \\ \text{a } P \\ \hline (P \wedge Q) \rightarrow P \end{array}$$

✓ v \wedge Elim
✓ v \rightarrow Intro

8.38.2

$$\begin{array}{|l} \text{a} \\ \hline \text{a } \neg(P \wedge Q) \vee P \end{array}$$

8.38.2

$$\begin{array}{|l} \text{a} \\ \hline \text{a } P \vee \neg P \\ \text{v } P \\ \text{a } \neg(P \wedge Q) \vee P \\ \text{v } \neg P \\ \hline \text{a } \neg(P \wedge Q) \vee P \\ \text{a } \neg(P \wedge Q) \vee P \end{array}$$

✓ v Taut Con
✓ v \vee Intro
✓ v \vee Elim

8.38.2

$$\begin{array}{|l} \text{a} \\ \hline \text{a } P \vee \neg P \\ \text{v } P \\ \text{a } \neg(P \wedge Q) \vee P \\ \text{v } \neg P \\ \hline \text{a } \neg(P \wedge Q) \\ \text{a } \neg(P \wedge Q) \vee P \\ \text{a } \neg(P \wedge Q) \vee P \end{array}$$

✓ v Taut Con
✓ v \vee Intro
✓ v \vee Intro
✓ v \vee Elim

8.38.2

$$\begin{array}{|l} \text{a} \\ \hline \text{a } P \vee \neg P \\ \text{v } P \\ \text{a } \neg(P \wedge Q) \vee P \\ \text{v } \neg P \\ \text{v } P \wedge Q \\ \text{a } P \\ \text{a } \perp \\ \text{a } \neg(P \wedge Q) \\ \text{a } \neg(P \wedge Q) \vee P \\ \text{a } \neg(P \wedge Q) \vee P \end{array}$$

✓ v Taut Con
✓ v \vee Intro
✓ v \wedge Elim
✓ \perp Intro
✓ v \neg Intro
✓ v \vee Intro
✓ v \vee Elim

8.29

$\begin{array}{l} P \rightarrow Q \\ P \vee \neg P \\ \quad P \\ \quad Q \\ \quad \neg P \vee Q \\ \quad \neg P \\ \quad \neg P \vee Q \\ \quad \neg P \vee Q \\ \quad \neg P \vee Q \\ \quad \quad P \\ \quad \quad \neg P \\ \quad \quad \perp \\ \quad \quad Q \\ \quad \quad Q \\ \quad \quad Q \\ P \rightarrow Q \end{array}$	<ul style="list-style-type: none"> ✓ Taut Con ✓ \rightarrow Elim ✓ \vee Intro ✓ \vee Intro ✓ \vee Elim ✓ \perp Intro ✓ \perp Elim ✓ \vee Elim ✓ \rightarrow Intro ✓ \leftrightarrow Intro
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$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

8.29

$\begin{array}{l} P \rightarrow Q \\ \neg P \vee Q \\ \neg P \vee Q \\ P \rightarrow Q \end{array}$	<ul style="list-style-type: none"> ✓ \leftrightarrow Intro
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$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

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