Section problems

Discussed: 17 January 2012 **CS103,Winter 2011-2012**

Problem 1:

Prove that $(\mathsf{P} \vee \mathsf{Q}) \to \mathsf{R} \equiv (\mathsf{P} \to \mathsf{R}) \wedge (\mathsf{Q} \to \mathsf{R})$ using (1) a truth table, and (2) an equational proof, using Boolean identities.

Solution:

Truth table:

P	Q	R	$(\mathsf{P} \vee \mathsf{Q})$	$\to$	R	$(\mathsf{P} \to \mathsf{R})$	$\wedge$	$(\mathsf{Q} \to \mathsf{R})$
T	**T**	**T**	**T**	**T**		**T**	**T**	**T**
T	**T**	**F**	**T**	**F**		**F**	**F**	**F**
T	**F**	**T**	**T**	**T**		**T**	**T**	**T**
T	**F**	**F**	**T**	**F**		**F**	**F**	**T**
F	**T**	**T**	**T**	**T**		**T**	**T**	**T**
F	**T**	**F**	**T**	**F**		**T**	**F**	**F**
F	**F**	**T**	**F**	**T**		**T**	**T**	**T**
F	**F**	**F**	**F**	**T**		**T**	**T**	**T**

Equational proof:

$$
\begin{aligned}
(\mathsf{P} \vee \mathsf{Q}) \to \mathsf{R} &\equiv \neg(\mathsf{P} \vee \mathsf{Q}) \vee \mathsf{R} && \text{implies-or} \\
&\equiv (\neg\mathsf{P} \wedge \neg\mathsf{Q}) \vee \mathsf{R} && \text{de Morgan} \\
&\equiv (\neg\mathsf{P} \vee \mathsf{R}) \wedge (\neg\mathsf{Q} \vee \mathsf{R}) && \text{distrib } \vee \text{ over } \wedge \\
&\equiv (\mathsf{P} \to \mathsf{R}) \wedge (\mathsf{Q} \to \mathsf{R}) && \text{implies-or}
\end{aligned}
$$

Problem 2: Consider the following formula:

$$\neg(P \vee Q) \vee \neg(R \vee \neg(S \vee \neg P))$$

a). Convert to a logically equivalent conjunctive normal form, using Boolean identities.

We did not do this in great detail, but here is the outline of the solution:

Step 1: Convert it to Negation Normal form using de Morgan's laws to get

$$(\neg P \wedge \neg Q) \vee (\neg R \wedge (S \vee \neg P))$$

Step 2: Apply Distributive laws repeatedly until we get CNF.

b). Convert to an equisatisfiable CNF using the "labeling" trick discussed in the lecture.

Solution:

Labeling trick:

$$X_1 \leftrightarrow (P \vee Q)$$

$$X_2 \leftrightarrow (S \vee \neg P)$$

$$X_3 \leftrightarrow (R \vee X_2)$$

$$X_4 \leftrightarrow (\neg X_1 \vee \neg X_3)$$

$$X_4$$

Now we $\wedge$ all of them to get the required

$$(X_1 \leftrightarrow P \vee Q) \wedge (X_2 \leftrightarrow S \vee \neg P) \wedge (X_3 \leftrightarrow R \vee X_2) \wedge (X_4 \leftrightarrow \neg X_1 \vee \neg X_3) \wedge X_4$$

Alternatively, we can say:

$$X_1 \leftrightarrow (P \vee Q)$$

$$X_2 \leftrightarrow (S \vee \neg P)$$

$$X_3 \leftrightarrow (R \vee X_2)$$

$$\neg X_1 \vee \neg X_3$$

to get

$$(X_1 \leftrightarrow P \vee Q) \wedge (X_2 \leftrightarrow S \vee \neg P) \wedge (X_3 \leftrightarrow R \vee X_2) \wedge (\neg X_1 \vee \neg X_3)$$

The biconditionals can be converted to CNF easily, but for homework you do not need to expand them.

Problem 3: Translate this problem into propositional logic and solve it using truth tables or Boolean identities.

"You are visiting an obscure island, where every inhabitant is either a knight or a knave. Knights always tell the truth. Knaves always lie (they only tell falsehoods). Some of the inhabitans are werewolves and have the annoying habit of sometimes turning into wolves at night and devouring unlucky tourists. A werewolf can be either a knight or a knave. You run into three inhabitants, Alice, Bob, and Carol. Alice declares, 'I am a werewolf.' Bob states I am a werewolf.' Carol says 'At most one of us is a knight.'

Is it possible that Carol is a werewolf?

KA, KB, KC – Alice, Bob, Carol are knights, respectively.

WA, WB, WC – Alice, Bob, Carol are werewolves, respectively.

Slight Detour If Zach (a hypothetical person) makes a statement S and if Zach is a knight then the statement S must be true: $KZ \rightarrow S$ (1).

If Zach is not a knight, he is a knave and always lies, then

$\neg KZ \rightarrow \neg S \equiv \neg\neg S \rightarrow \neg\neg KZ$ (contrapositive) $\equiv S \rightarrow KZ$ (2).

From (1), (2), $KZ \leftrightarrow S$ (by definition).

Coming back to our problem,

Alice declares, 'I am a werewolf.' translates to: $KA \leftrightarrow WA$. (3)

Bob states I am a werewolf.' translates to: $KB \leftrightarrow WB$. (4)

Carol says 'At most one of us is a knight.' Let's now see how we can convert Carol's statement into logic. At most one of KA, KB, KC is true. From class notes, that can be written as

$$(\neg KA \wedge \neg KB) \vee (\neg KB \wedge \neg KC) \vee (\neg KA \wedge \neg KC)$$

The above statement is true iff Carol is a Knight. Thus Carol's declaration translates to the following biconditional:

$KC \leftrightarrow [(\neg KA \wedge \neg KB) \vee (\neg KB \wedge \neg KC) \vee (\neg KA \wedge \neg KC)]$ (5)

Carol is a werewolf: (We want to see if this statement is satisfiable given the above three statements).

WC. (6)

The logic problem is: Is the conjunction of the above four statements (3,4,5,6) satisfiable?

We don't have time to write out the truth table as it has 64 entries. But for your homework you will have only four, hence it will be easier for you to write out the truth table for the HW1.

Let's look for a satisfying assignment using case analysis (this is a technique that was not discussed in class - it is not necessary for the HW though):

Case 1: Suppose KC is True (i.e. Carol is a knight). from the biconditional in (5) the following statement must be true

$$(\neg KA \land \neg KB) \lor (\neg KB \land \neg KC) \lor (\neg KA \land \neg KC)$$

On Simplification,

$$(\neg KA \land \neg KB) \lor (\neg KB \land \mathbf{F}) \lor (\neg KA \land \mathbf{F})$$

$$(\neg KA \land \neg KB) \lor \mathbf{F} \lor \mathbf{F}$$

$$(\neg KA \land \neg KB)$$

Given $KA \leftrightarrow WA$ (3) and $KB \leftrightarrow WB$ (4), $\neg WA$ and $\neg WB$.

We get: Alice and Bob are knaves. Carol is a knight. Alice and Bob are not werewolves. Carol is free to be a werewolf.

Case 2: Suppose $\neg KC$ is True (i.e. Carol is not a knight). from the biconditional in (5) the following statement must be true

$$\neg[(\neg KA \land \neg KB) \lor (\neg KB \land \neg KC) \lor (\neg KA \land \neg KC)]$$

On Simplification

$$(KA \lor KB) \land (KB \lor KC) \land (KA \lor KC)$$

$$(KA \lor KB) \land (KB \lor \mathbf{F}) \land (KA \lor \mathbf{F})$$

$$(KA \lor KB) \land (KB) \land (KA)$$

$$KA \land KB$$

Again from (5) and (6) we have the following to be True

$$WA \wedge WB$$

In this case, Alice and Bob are Knights and Carol is not. Alice and Bob are werewolves. Again, WC can be True!

Thus, we found at least two satisfying assignments (for the six variables) in which Carol is a werewolf. Hence it is recommended that you stay away from her.