First Order Logic I

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Announcements

- Problem sessions: 7:00 – 8:30 PM Hewlett 101. Mondays (almost always), but Tuesdays after a Monday holiday.
- The first section will be next Tuesday, January 17th.
- The first homework is up on the web page. Start early so you have time to get help if you are confused. (It will help to look through the problems before the problem section.)

Outline

- Clarification of Sudoku
- Introduction to Quantifiers
- Well-formed Formulas
- Validity and Logical Consequence
- Identities
- Aristotelian Forms
- Multiple Quantifiers

Row/column/block constraints

In the last lecture, my explanation was unclear. I'll describe something a bit different from what my program did.

There are 9 row formulas, 9 column formulas, and 9 block formulas. Each formula is big, but conceptually simple.

Let's name the propositional variables $P_{ijk\ell m}$ where i, j, k, ℓ are the square coordinates as in the previous lecture (1 to 3), and m is the value in the square (1 to 9).

Whenever two propositional symbols are in the same row but different columns ($k_1 \neq k_2$ or $\ell_1 \neq \ell_2$), there is a formula saying that two different squares cannot both be m.

$$\neg P_{ijk_1\ell_1 m} \vee \neg P_{ijk_2\ell_2 m}$$

A row formula is the conjunction of all formulas like this for all i, j, k_1, k_2, ℓ_1, ℓ_2 (in the range 1 to 3) and all m (in the range 1 to 9).

The column and block formulas are similar.

Predicate Logic

Predicate logic (also called "First-order logic") is more expressive than propositional logic ("more expressive" – we can say more things).

Predicate logic formulas can talk about properties of individual things and relationships between them.

Predicate logic has

- *Individual variables* that can refer to individuals (e.g. x);
- *Predicates* which refer to properties of individuals or relationships among them (e.g. Big(x), Taller(x, y), $x = y$);
- and *function symbols* which describe functions that map groups of individuals to other individuals (e.g., height(x), maximum(x, y)).

Other symbols

In some sense, there are many dialects of predicate logic.

For example, in integer arithmetic, we might have functions $+$, $-$, and $\times$ and $\geq$.

These are *interpreted* symbols because we all agree on what they mean.

In arithmetic, we would also have infinitely many constants such as 0, 1, -1, etc.

(I didn't mention "constants" in predicate logic, because they can be consider functions with no arguments.)

Sometimes, a predicate logic can have "uninterpreted" functions and predicates.

That means that they have no particular meaning. They are still useful, because logic can be used to define their properties.

Identity

Other symbols may vary between "dialects" of predicate logic, but $=$ (called *identity* or *equality*) is always there, and it has certain properties. For any x, y, and z:

- **reflexivity:** $x = x$
- **symmetry:** If $x = y$ then $y = x$
- **transitivity:** If $x = y$ and $y = z$, then $x = z$.

Equality has fourth special property:

substition: If $x = y$, and f is any function, then $f(x) = f(y)$.

Quantifiers

"All men are mortal" cannot be expressed with the logical language we have defined up to now.

It requires a new concept: the *quantifier*.

First order logic has two quantifiers: $\forall$ ("every") and $\exists$ ("exists").

$\forall x\ (\mathrm{Man}(x) \rightarrow \mathrm{Mortal}(x))$ For every x, if x is a Man then x is Mortal.

Other key phrases: "For all", "Each", "Any" (but be careful about "any"; it can sometimes be interpreted as "some"), etc.

$\exists x\ (\mathrm{Man}(x) \wedge \mathrm{Tall}(x))$ There exists an x such that x is a Man and x is Tall.

Other key phrases: "For some", "there is some", "there is a/an", etc.

Quantifier Intuition

$\forall$ is "AND over everything", $\exists$ is "OR over everything."

E.g., if we're talking about non-negative integers:

$\forall x\ P(x) = P(0) \wedge P(1) \wedge \ldots \wedge P(n) \wedge \ldots$

$\exists x\ P(x) = P(0) \vee P(1) \vee \ldots \vee P(n) \vee \ldots$

Quantification is *more powerful* than $\wedge$ and $\vee$ because it can express AND/OR over *infinite* sets.

Variables

Quantifiers range over *variables*.

Variables represent individuals, as do individual constants, but the value of the variable is not fixed.

Notation: t, u, v, w, x, y, z will generally be variables (subscripts are ok, e.g., x_{31})

Variables can appear wherever individual constants can appear, and (of course) right after quantifiers.

Well-formed formulas

Terms can contain function symbols, individual constants, and *variables*.

E.g., a, x, $f(a)$, $f(a, x)$

An *Atomic formula* is a predicate with terms as arguments: P, $SameRow(a, f(b, g(x)))$, $f(a) = g(x, b)$.

First-order formulas are constructed using Boolean operators and quantifiers:

- An atomic formula is a formula.
- $\neg P$ is a formula if P is a formula.
- $P_1 \wedge \ldots \wedge P_n$ is a formula if $P_1, \ldots P_n$ are formulas.
- $P_1 \vee \ldots \vee P_n$ is a formula if $P_1, \ldots P_n$ are formulas.
- $P \rightarrow Q$ is a formula if P and Q are formulas.
- $P \leftrightarrow Q$ is a formula if P and Q are formulas.
- $\forall \nu\ P$ is a formula if P is a formula and ν is a variable.
- $\exists \nu\ P$ is a formula if P is a formula and ν is a variable.

Free and Bound Variables

ν is a *bound variable* in $\forall \nu\ P$ and $\exists \nu\ P$.

(A bound variable is like a local variable in a program).

A *free variable* is an unbound variable.

Examples:

x is bound in $\forall x\ P(x)$
x is free in $P(x)$
x is bound and y is free in $\forall x\ (x < y)$.
x is both bound and free in $P(x) \wedge \forall x\ Q(x)$.

A *sentence* is a formula with no free variables.

Semantics of Quantifiers

The *domain of discourse* is the collection of all objects in the "world." This depends on how the logic formulas are to be used.

We assume the domain of discourse is non-empty.

One domain of discourse might be the integers.

Quantifiers are not truth-functional:

> *To understand* $\forall x\ P(x)$*, we can't just ask whether* $P(x)$ *is true (we don't necessarily know whether it's true because we don't know what* x *is).*

Let's assume every object in the domain of discourse has a name.

Then $\forall x\ P(x)$ is true if $P(a)$ is true for *every* name a that can be substituted for x in $P(x)$.

Similarly, $\exists x\ P(x)$ is true if $P(a)$ is true for *some* a substituted for x.

Validity

A formula is valid if it is always true.

Validity might be based purely on logical reasoning, independently of the interpretation of functions and predicates:

Examples:

$a = a$ $\qquad a = b \lor a \neq b$

$\forall x\ (x = a \lor x \neq a) \qquad \forall x\ (P(x) \lor \neg P(x))$

Validity could also depend on interpreted symbols:

$1 + 1 = 2 \qquad x \times (y + z) = x \times y + x \times z$

$\forall x\ (x + 1 > x) \qquad x \times x \geq 0$

Logical Consequence

In proofs, we have a set of *premises* (logical formulas that are assumed to be true). By logical reasoning from the premises, we arrive at a *conclusion*.

A first-order sentence is a *logical consequence* of first-order premises if, in any world where the premises hold, the conclusion holds.

Examples:

$a = c$ is a consequence of the premises $a = b$ and $b = c$, because $=$ is transitive.

$x \times y > 0$ is a consequence of the premises $x > 0$ and $y > 0$.

Counterexample

A first-order sentence is *not* valid iff there exists a world (called a *counterexample*) in which it is not true.

To define a counterexample, you must define:

- the domain of discourse
- the meanings of the individual constants that appear in the formula.
- the meanings of the functions. *Make sure the function is well-defined (everything must map to exactly one value).*
- the meanings of the predicates.

An Example of a Counterexample

Show that $\forall x\ R(x, f(a))$ is not a logical consequence of $\forall x\ R(x, f(x))$.

- Domain of discourse: The numbers 0 and 1.
- a is 0
- $f(0) = 0$ and $f(1) = 1$, so $\forall x\ f(x) = x$. (is it well-defined?)
- R is $\leq$. (R does not have to be a well-known relation like $\leq$. We could have defined it explicitly: $R(0, 0)$, $R(1, 1)$, and $R(0, 1)$ are true; $R(1, 0)$ is false.)

This satisfies the premise $\forall x\ R(x, f(x)) (\Leftrightarrow \forall x\ x \leq x)$ ($\leq$ is *reflexive*).

But falsifies the conclusion $\forall x\ R(x, f(a)) (\Leftrightarrow \forall x\ x \leq 0)$, since $1 \not\leq 0$.

The Aristotelian Forms

This is not just ancient history or philosophy. These are the among the most common statements in FOL.

Idea: Usually, quantifiers are *restricted* to a predicate (P here).

"All P's are Q's"	$\forall x\ (P(x) \rightarrow Q(x))$
"Some P's are Q's"	$\exists x\ (P(x) \wedge Q(x))$
"No P's are Q's"	$\forall x\ (P(x) \rightarrow \neg Q(x))$
"Some P's are not Q's"	$\exists x\ (P(x) \wedge \neg Q(x))$

Important: Note the asymmetry between $\rightarrow$ in the $\forall$ case and $\wedge$ in the $\exists$ case.

A very common mistake is to say: $\exists x\ (P(x) \rightarrow Q(x))$. When you see something that looks like this, it is almost always an error – triple-check it.

Vacuously True Sentences

This is a subtle but important point.

$\forall x\ (P(x) \rightarrow Q(x))$ is true if $P(x)$ is never true.

When this occurs, the sentence is said to be *vacuously true*.

Example: $\forall x\ (x > x \rightarrow x = x + 1))$ is valid!

Example: In a world with only object $\forall x\ (\forall y\ [(x \neq y) \rightarrow x = y])$ is true!