Section problems

Discussed: 17 January 2012 **CS103,Winter 2011-2012**

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Problem 1: $\sqrt{2}$ is irrational.

Solution:

We prove by contradiction. Suppose $\sqrt{(2)}$ were rational. Then by a theorem from Monday's lecture, there exist integers p and q with no common factors such that $\sqrt{2} = p/q$ [This used existential instantiation, since we just gave the names p and q to these integers that were shown to exist in the theorem of the lecture.] By simple algebra, $2 = \frac{p^2}{q^2}$ and so $2q^2 = p^2$. But then p^2 is even, so, by another theorem of Monday's lecture p is also even. So there exists some p' such that $p = 2p'$. But then we have $q^2 = 2p'^2$, so, by the same reasoning, q is also even. But that contradicts the fact the p and q have no common factors, so the initial assumption that $\sqrt{2}$ is rational must be false.

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Problem 2: There exists unique x such that P(x) . Write using normal quantifiers

Solution: $\exists! x P(x)$, using normal form:

- $\exists x P(x)$
- $\exists x \forall y (P(y) \rightarrow x = y)$

The above two statements should hold simultaneously. Thus we have
$\exists! x P(x) \equiv (\exists x P(x)\ ) \wedge (\exists x \forall y (P(y) \rightarrow x = y))$

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Problem 3: Convert to PNF
$\forall x((\neg \forall x P(x)) \vee Q(x)) \wedge \forall x R(x)$

Identities used in Solution:
Distributive law ($\vee$ over $\wedge$)
$(\forall x P(x)) \vee Q \equiv \forall x(P(x) \vee Q)$
Distributive law ($\wedge$ over $\vee$)
$(\exists x P(x)) \wedge Q \equiv \exists x(P(x) \wedge Q)$
Idempotent laws
$(\forall x P(x)) \wedge Q \equiv \forall x(P(x) \wedge Q)$
$(\exists x P(x)) \vee Q \equiv \forall x(P(x) \vee Q)$
Associate laws
$\forall x(P(x) \wedge \forall y R(y)) \equiv \forall x \forall y(P(x) \wedge R(y)) \equiv \forall x(P(x) \wedge R(x))$
$\exists x(P(x) \vee \exists y R(y)) \equiv \exists x \exists y(P(x) \vee R(y)) \equiv \exists x(P(x) \vee R(x))$

Solution:

$$\begin{array}{ll}
\forall x((\neg \forall x P(x)) \vee Q(x)) \wedge \forall x R(x) & \\
\forall x((\neg \forall y P(y)) \vee Q(x)) \wedge \forall x R(x) & \text{(Change of variable)} \\
\forall x((\exists y \neg P(y)) \vee Q(x)) \wedge \forall x R(x) & \text{(Demorgan's Law)} \\
\forall x\{((\exists y \neg P(y)) \vee Q(x)) \wedge R(x)\} & \text{(Associativity)} \\
\forall x \exists y\{((\neg P(y)) \vee Q(x)) \wedge R(x)\} & \text{(Distribute } \wedge \text{ over } \vee\text{)}
\end{array}$$

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Problem 4:
From Rosen. Section 1.5 Prob 33 (or 3 in some editions)
Prove that x + y = max(x,y) + min (x,y) by cases.

Cases	$\max(x, y)$	$\min(x, y)$	$max(x, y) + min(x, y)$
$x \leq y$	y	x	$x + y$
$y \leq x$	x	y	$x + y$

So all cases of real numbers x, y, we have x + y = max(x,y) + min (x,y).