Homework 2

Due: Jan 27, 2012 **CS103,Winter 2011-2012**

Correction (1/21/2012): specified non-negative integers in problem 8.

Correction (1/22/2012): Problem 11 is solved in Rosen Chapter 1. Don't bother doing it.

Correction (1/23/2012): Problem 9 should specify that x is non-zero.

For this homework, write all informal proofs according to the guidelines given in lectures up to lecture 5. In particular, when using a universal generalization, explicitly say "Let n be an arbitrary integer . . . ", etc. Later, we will relax this rule, but please do it for now.

Problem 1 (Rosen, section 1.3, problem 36): Find a counterexample, if possible, to these universally quantified statements, where the domain for all varialbes consists of all real numbers.

a. $\forall \mathsf{x}\ (\mathsf{x}^2 \neq \mathsf{x})$

b. $\forall \mathsf{x}\ (\mathsf{x}^2 \neq 2)$

c. $\forall \mathsf{x}\ (|\mathsf{x}| > 0)$

Problem 2 (Rosen, section 1.3, problem 50): Show that $\forall \mathsf{x}\ \mathsf{P}(\mathsf{x}) \lor \forall \mathsf{x}\ \mathsf{Q}(\mathsf{x})$ and $\forall \mathsf{x}\ (\mathsf{P}(\mathsf{x}) \lor \mathsf{Q}(\mathsf{x}))$ are not logically equivalent.

Problem 3 (Rosen, section 1.4, problem 40). Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a. $\forall \mathsf{x} \exists \mathsf{y}\ (\mathsf{x} = 1/\mathsf{y})$

b. $\forall \mathsf{x} \exists \mathsf{y}\ (\mathsf{y}^2 - \mathsf{x} < 100)$

c. $\forall \mathsf{x} \forall \mathsf{y}\ (\mathsf{x}^2 \neq \mathsf{y}^3)$

Problem 4 (Rosen, section 1.4, problem 50): A statement is in **prenex normal form(PNF)** if and only if it is of the form $Q_1 x_1 Q_2 x_2 \cdots Q_k x_k\ P(x_1, x_2, \ldots, x_k)$. Put these statements in prenex normal form.

Hint: Use logical equivalences from Tables 6 and 7 in Section 1.2, Table 2 in Section 1.3, Example 19 in Rosen section 1.3, Exercises 45 and 46 in Section 1.3, and Exercises 48 and 49 in Rosen section 1.4.

a. $\exists x\ P(x) \lor \exists x\ Q(x) \lor A$, where A is a proposition not involving any quantifiers.

b. $\neg(\forall x\ P(x) \lor \forall x\ Q(x))$

c. $\exists x\ P(x) \rightarrow \exists x\ Q(x)$

Problem 5 Write a proof in System F of following $\forall x\ (P(x) \land Q(x)) \leftrightarrow (\forall x\ P(x) \land \forall y\ Q(y))$

Problem 6 Here is a "proof" of the "theorem":

$$\forall x\ \exists y\ R(x, y) \rightarrow \exists y\ \forall x\ R(x, y)$$

which is unfortunate, because the "theorem" is not actually valid.

1.	$\forall x\ \exists y\ R(x, y)$	
2.	$\boxed{a}$	
3.	$\exists y\ R(a, y)$	$\forall$ Elim: 1
4.	$\boxed{b}\ R(a, b)$	
5.	$R(a, b)$	Reit 4
6.	$R(a, b)$	$\exists$ Elim 4–5,3
7.	$\forall x\ R(x, b)$	$\forall$ Intro 3–6
8.	$\exists y\ \forall x\ R(x, y)$	$\exists$ Intro 7

a. Describe a counter-example to the "theorem."

b. What rule was misapplied (based on the descriptions of the rules in the lecture)?

Problem 7 Consider the following statement: *There is a number between every pair of non-equal numbers.* (In this context, "between" two numbers also means not equal to either number).

a. Write the statement in first order logic using standard symbols from arithmetic operations and relations (in addition to the usual logical symbols).

b. Is it true for the integers? Prove it (show the existence of the "between" number by constructing it), or give a counterexample.

c. Is it true for the rational numbers?

   Prove it (show the existence of the "between" number by constructing it), or give a counterexample.

Rational numbers are numbers that can be written as p/q where p and q are both integers.

Problem 8 Write an informal proof by contradiction of the following theorem, following the guidelines in the lectures: The only consecutive non-negative integers satisfying $a^2 + b^2 = c^2$ are $a = 3$, $b = 4$, and $c = 5$ (by "consecutive", we mean $c = b + 1 = a + 2$).

Problem 9 Prove by contradiction that, for all real numbers x and y, if x is a non-zero rational number and and y is irrational, the xy is also irrational.

Problem 10 Prove by cases that $\min(x, \min(y, z)) = \min(\min(x, y), z)$, for all real numbers x, y, and z.

Problem 11 *[I overlooked the fact that this is solved as an example in the reader. Don't do it.]* Prove that the square of every integer has a last digit of 0, 1, 4, 5, 6, or 9.