Exercise 1 (Darcy equations) Let Ω be a smooth bounded domain of $\mathbb{R}^d$, and $\boldsymbol{n}$ the outward unit normal on $\partial\Omega$. Let $\boldsymbol{f} \in (L^2(\Omega))^d$, $g \in H^{1/2}(\partial\Omega)$ and $k \in L^2(\Omega)$. This problem deals with two different weak formulations of the Darcy equations.

Preliminary remarks

- *Property 1:* Let $f \in L^2(\Omega)$ (strictly, it is sufficient that $f \in L^1_{\text{loc}}(\Omega)$). Then, if

  $$\forall \phi \in \mathcal{C}_0^\infty(\Omega) \quad \int_\Omega f\phi = 0,$$

  $f = 0$ almost everywhere in Ω. One can write $f = 0$.

  This result can be extended to vector valued functions $\boldsymbol{f} \in (L^2(\Omega))^d$.

- *Property 2:* Let $f \in L^2(\Omega)$. If $\forall q \in L^2(\Omega)$

  $$\int_\Omega fq = 0$$

  then $f = 0$ almost everywhere in Ω.

  Proof: Consider $q = f$. Then

  $$\|f\|^2_{L^2(\Omega)} = \int_\Omega f^2 = 0.$$

- About the trace of $H(div;\Omega)$ functions.

  It is well-known that one can define the trace of $H^1(\Omega)$ functions on $\partial\Omega$ whereas it is not possible to define the trace of a $L^2(\Omega)$ function. The space of traces of $H^1(\Omega)$ functions is $H^{1/2}(\partial\Omega)$ (which in particular contains $L^2(\partial\Omega)$). A $H(div;\Omega)$ function is less regular than a $H^1(\Omega)^d$ function, nevertheless a notion of *normal trace* on $\partial\Omega$ can still be defined for $H(div;\Omega)$. The definition relies on the following equality: Let $\boldsymbol{u}$ and ϕ be smooth functions,

  $$\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n}\phi = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\nabla}\phi + \int_\Omega \phi \operatorname{div} \boldsymbol{u}.$$

  Note that the right-hand side makes sense for a function $\phi \in H^1(\Omega)$. In such a case, in the left-hand side, $\boldsymbol{u} \cdot \boldsymbol{n}$ acts on a $H^{1/2}(\partial\Omega)$ function. It is therefore natural to define, by density, the normal trace $\boldsymbol{u} \cdot \boldsymbol{n}$ of any function $\boldsymbol{u} \in H(div;\Omega)$ as the linear form acting on $H^{1/2}(\partial\Omega)$. Thus $\boldsymbol{u}\cdot\boldsymbol{n}$ is an element of the dual of $H^{1/2}(\partial\Omega)$, which is denoted by $H^{-1/2}(\partial\Omega)$.

In conclusion, if $\boldsymbol{u} \in H(div;\Omega)$, then $\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ is defined by:

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}, \phi\rangle = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{\nabla}\phi + \int_\Omega \phi \,\mathrm{div}\,\boldsymbol{u}, \quad \forall\phi \in H^{1/2}(\partial\Omega).$$

Thus in particular

$$\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1/2}} \leq C\|\boldsymbol{u}\|_{H(div;\Omega)}.$$

Let $\boldsymbol{v} \in H(div;\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Strictly speaking, it is not correct to write

$$\int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{n} g.$$

One should rather write $\langle \boldsymbol{v}\cdot\boldsymbol{n}, g\rangle$. But we will accept the integral form as an abuse of notations.

For those of you who are not familiar with these notions and spaces, it is perfectly acceptable for this course (homeworks and exams) to do formal computations assuming that all the functions are regular enough to give a sense to the integrations by parts. The most important thing is to be able to derive (even formally) a PDE and the boundary conditions from a variational formulation.

Part 1

Consider the problem: Search for $(\boldsymbol{u}, p) \in H(\mathrm{div};\Omega)\times L^2(\Omega)$ such that for all $(\boldsymbol{v}, q) \in H(\mathrm{div};\Omega)\times L^2(\Omega)$

$$(P)\left\{\begin{array}{rcl} \displaystyle\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} - \int_\Omega p\,\mathrm{div}\,\boldsymbol{v} &=& \displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\partial\Omega} g\boldsymbol{v}\cdot\boldsymbol{n} \\ \displaystyle\int_\Omega q\,\mathrm{div}\,\boldsymbol{u} &=& \displaystyle\int_\Omega kq \end{array}\right.$$

1) What are the partial differential equations and the boundary conditions corresponding to this problem? Are the boundary conditions enforced naturally or essentially ?

$\forall \boldsymbol{v} \in \mathcal{C}_0^\infty(\Omega)\bigcap H(div;\Omega)$

$$-\int_\Omega p\,\mathrm{div}\,\boldsymbol{v} = \int_\Omega (\boldsymbol{f}-\boldsymbol{u})\cdot\boldsymbol{v}.$$

p is therefore weakly differentiable since

$$\left|\int_\Omega p \operatorname{div} \boldsymbol{v}\right| \leq C\|\boldsymbol{v}\|_{L^2(\Omega)}.$$

Defining weak partial derivatives $\frac{\partial p}{\partial x_i},\ i = 1, ..., d,$

$$\int_\Omega (\boldsymbol{u} + \boldsymbol{\nabla} p - \boldsymbol{f}) \cdot \boldsymbol{v} = 0,$$

Using Property 1, $\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f}$ and $\boldsymbol{\nabla} p \in L^2(\Omega)$. Therefore, $p \in H^1(\Omega)$ and p has a trace on $\partial\Omega$.

Choosing now $\boldsymbol{v} \in H(div; \Omega)$,

$$-\int_{\partial\Omega} p\boldsymbol{v} \cdot \boldsymbol{n} = \int_{\partial\Omega} g\boldsymbol{v} \cdot \boldsymbol{n},$$

thus

$$p|_{\partial\Omega} = -g.$$

This Dirichlet boundary condition is enforced naturally.

$\forall q \in L^2(\Omega)$

$$\int_\Omega q(\operatorname{div} \boldsymbol{u} - k) = 0.$$

Using Property 2, this proves that

$$\operatorname{div} \boldsymbol{u} = k.$$

In fine, the PDE and BC associated with the problem are

$$\left\{\begin{array}{rcl} \boldsymbol{u} + \boldsymbol{\nabla} p & = & \boldsymbol{f} \ \text{ in } \Omega \\ \operatorname{div} \boldsymbol{u} & = & k \ \text{ in } \Omega \\ p & = & -g \ \text{ on } \partial\Omega \end{array}\right.$$

2) Let $q \in L^2(\Omega)$. Prove there exists a unique $\Phi \in H_0^1(\Omega)$ such that

$$\int_\Omega \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\psi = \int_\Omega q\psi, \qquad \textbf{for all } \psi \in H_0^1(\Omega).$$

Set $\boldsymbol{v} = \boldsymbol{\nabla}\Phi$. Give an estimate (upper bound) of $\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)}$.

Existence and uniqueness of Φ results from the Lax-Milgram theorem.

Moreover, using Poincare's inequality

$$\int_\Omega |\boldsymbol{\nabla}\Phi|^2 = \int_\Omega q\Phi \leq \|q\|_{L^2(\Omega)}\|\Phi\|_{L^2(\Omega)} \leq C_\Omega\|q\|_{L^2(\Omega)}\|\boldsymbol{\nabla}\Phi\|_{L^2(\Omega)}.$$

Thus
$$\|\boldsymbol{v}\|_{L^2(\Omega)} = \|\boldsymbol{\nabla}\Phi\|_{L^2(\Omega)} \le C_\Omega \|q\|_{L^2(\Omega)}.$$
From $\operatorname{div}\boldsymbol{v} = -q$,
$$\|\operatorname{div}\boldsymbol{v}\|_{L^2(\Omega)} \le \|q\|_{L^2(\Omega)}$$
and
$$\|\boldsymbol{v}\|_{H(div;\Omega)} \le \tilde{C}\|q\|_{L^2(\Omega)},$$
with $\tilde{C} = \sqrt{1 + \boldsymbol{C}_\Omega^2}$.

3) Deduce from the previous question that there exists $\beta > 0$ such that for all $q \in L^2(\Omega)$,
$$\sup_{\boldsymbol{w}\in H(\mathrm{div};\Omega)} \frac{-\displaystyle\int_\Omega q \operatorname{div}\boldsymbol{w}}{\|\boldsymbol{w}\|_{H(\mathrm{div};\Omega)}} \ge \beta\|q\|_{L^2(\Omega)}. \tag{1}$$

Let $q \in L^2(\Omega)$ and $\boldsymbol{v} = \boldsymbol{\nabla}\Phi$ be given as in question 2. Then
$$\int_\Omega -q \operatorname{div}\boldsymbol{v} = \int_\Omega q^2$$
and
$$\frac{-\int_\Omega q \operatorname{div}\boldsymbol{v}}{\|q\|_{L^2(\Omega)}} = \|q\|_{L^2(\Omega)} \ge \frac{1}{\tilde{C}}\|\boldsymbol{v}\|_{H(div;\Omega)}.$$
Thus
$$\sup_{\boldsymbol{w}\in H(\mathrm{div};\Omega)} \frac{-\displaystyle\int_\Omega q \operatorname{div}\boldsymbol{w}}{\|\boldsymbol{w}\|_{H(\mathrm{div};\Omega)}} \ge \frac{-\displaystyle\int_\Omega q \operatorname{div}\boldsymbol{v}}{\|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)}} \ge \frac{1}{\tilde{C}}\|q\|_{L^2(\Omega)}.$$

4) Let $q \in L^2(\Omega)$. Set $\alpha = \frac{1}{\mathbf{meas}(\Omega)}\int_\Omega q$. Construct $\boldsymbol{v}_\alpha \in (H^1(\Omega))^d$ such that $\operatorname{div}\boldsymbol{v}_\alpha = \alpha$. Prove that the operator div is surjective from $(H^1(\Omega))^d$ onto $L^2(\Omega)$.

Hint: Use the following result from the course: the operator div ***is surjective from*** $(H_0^1(\Omega))^d$ ***onto*** $L_0^2(\Omega)$.

Let $\alpha \in \mathbb{R}$. Let's consider the direction $\mathbf{e}_1$, and define
$$\boldsymbol{v}_\alpha = \alpha x_1 \mathbf{e}_1.$$
Then $\operatorname{div}\boldsymbol{v}_\alpha = \alpha$.

Let $q \in L^2(\Omega)$. Defining
$$q_0 = q - \frac{1}{|\Omega|}\int_\Omega q \in L_0^2(\Omega).$$

Using the hint, $\exists \boldsymbol{v}_0 \in (H_0^1(\Omega))^d$ such that

$$\operatorname{div} \boldsymbol{v}_0 = q_0 = q - \frac{1}{|\Omega|}\int_\Omega q = q - \operatorname{div} \boldsymbol{v}_\alpha.$$

Letting $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_\alpha$, $\operatorname{div} \boldsymbol{v} = q$ with $\boldsymbol{v} \in (H^1(\Omega))^d$, thusd div is surjective from $(H^1(\Omega))^d$ onto $L^2(\Omega)$.

5) Deduce from the previous question another proof of (1).

Let $q \in L^2(\Omega)$. Using question 4, there exists $\boldsymbol{v} \in (H^1(\Omega))^d$ such that $\operatorname{div} \boldsymbol{v} = q$. Therefore, $\boldsymbol{v} \in H(div;\Omega)$. Since the application

$$\begin{array}{ccc} H(div;\Omega) & \rightarrow & L^2(\Omega) \\ \boldsymbol{v} & \longmapsto & \operatorname{div} \boldsymbol{v} \end{array}$$

is continuous, the Brezzi theorem states that the inf-sup condition (1) is satisfied.

6) Prove that problem (P) is well-posed.

Defining $a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}$, $a(\boldsymbol{u},\boldsymbol{v})$ is coercive in $V = \{\boldsymbol{v} \in H(div;\Omega),\ \operatorname{div} \boldsymbol{v} = 0\}$.

Furthermore, $b(\boldsymbol{v}, q)$ satisfies the inf-sup condition in $X \times M = H(div;\Omega) \times L^2(\Omega)$.

Defining

$$< T, \boldsymbol{v} >= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\partial\Omega} g\boldsymbol{v}\cdot\boldsymbol{n},$$

and

$$< S, q >= \int_\Omega kq,$$

$T \in X'$ since

$$| < T, \boldsymbol{v} > | \leq \|f\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{H^{-1/2}(\partial\Omega)}.$$

$S \in M'$ since

$$| < S, q > | \leq \|k\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}.$$

This shows that the problem is well-posed.

Part 2

7) Assume that $\int_\Omega k + \int_{\partial\Omega} g = 0$ and define the space $H^1_{\int=0}(\Omega) = \{q \in H^1(\Omega), \int_\Omega q = 0\}$ equipped with the norm $\|\boldsymbol{\nabla}\cdot\|_{L^2(\Omega)}$. Consider

the problem: search for $(\boldsymbol{u}, p) \in (L^2(\Omega))^d \times H^1_{\int=0}(\Omega)$ such that for all $(\boldsymbol{v}, q) \in (L^2(\Omega))^d \times H^1_{\int=0}(\Omega)$

$$(Q)\left\{\begin{array}{rcl} \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{\nabla} p\cdot\boldsymbol{v} &=& \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \\ -\int_\Omega \boldsymbol{\nabla} q\cdot\boldsymbol{u} &=& \int_\Omega kq + \int_{\partial\Omega} gq \end{array}\right.$$

What are the partial differential equations and the boundary conditions corresponding to this problem? Are the boundary conditions enforced naturally or essentially ?

Hint: Prove that u $\in H(div;\Omega)$ and admit the fact that every function in $H(div,\Omega)$ has a well-defined normal trace on $\partial\Omega$

Let $\boldsymbol{v} \in \mathcal{C}_0^\infty(\Omega)$.

$$\int_\Omega (\boldsymbol{u} + \boldsymbol{\nabla} p - \boldsymbol{f})\cdot\boldsymbol{v} = 0.$$

Using Property 1, $\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f}$.

Let $q \in \mathcal{C}_0^\infty(\Omega) \bigcap H^1_{\int=0}(\Omega)$. Then

$$\int_\Omega \boldsymbol{\nabla} q\cdot\boldsymbol{u} = \int_\Omega kq.$$

This shows that u is weakly differentiable in terms of its divergence, as

$$\left|\int_\Omega \boldsymbol{\nabla} q\cdot\boldsymbol{u}\right| \leq \|k\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}.$$

Then, one can write

$$-\int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n}q + \int_\Omega q \operatorname{div}\boldsymbol{u} = \int_\Omega kq$$

and, as $q = 0$ on $\partial\Omega$,

$$\int_\Omega q \operatorname{div}\boldsymbol{u} = \int_\Omega kq.$$

Using Property 1,

$$\operatorname{div}\boldsymbol{u} = k + C \text{ in } \Omega.$$

Furthermore this shows that $\operatorname{div}\boldsymbol{u} \in L^2(\Omega)$ and thus $\boldsymbol{u} \in H(\mathrm{div};\Omega)$.

The hint shows that $\boldsymbol{u}\cdot\boldsymbol{n}$ is well-defined on $\partial\Omega$.

Then, letting $q \in H^1_{\int=0}(\Omega)$,

$$\int_{\partial\Omega} gq + \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n}q = C\int_\Omega q = 0.$$

Then $-\boldsymbol{u} \cdot \boldsymbol{n} = g$ in $\partial\Omega$. This is a natural boundary condition.

Going back to $\operatorname{div} \boldsymbol{u} = k + C$,

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} = \int_{\Omega} k + |\Omega| C,$$

that is

$$-\int_{\partial\Omega} g - \int_{\Omega} k = |\Omega| C.$$

Using the compatibility condition $\int_{\partial\Omega} g + \int_{\Omega} k = 0$,

$$|\Omega| C = 0 \Rightarrow C = 0,$$

and

$$\operatorname{div} \boldsymbol{u} = k.$$

The PDE and BC associated with the problem are

$$\left\{ \begin{array}{rcll} \boldsymbol{u} + \boldsymbol{\nabla} p & = & \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} & = & k & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} & = & -g & \text{on } \partial\Omega \end{array} \right.$$

8) Prove that problem (Q) is well-posed. $a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{v}$ is coercive on $L^2(\Omega)$.

Let $b(\boldsymbol{v}, q) = \int_{\Omega} \boldsymbol{\nabla} p \cdot \boldsymbol{v}$. Let $p \in H^1_{\int=0}$ and $\boldsymbol{v} = \boldsymbol{\nabla} p \in L^2(\Omega)$. Then

$$\frac{\int_{\Omega} \boldsymbol{\nabla} p \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|_{L^2(\Omega)}} = \frac{\int_{\Omega} |\boldsymbol{\nabla} p|^2}{\|\boldsymbol{\nabla} p\|_{L^2(\Omega)}} = \|\boldsymbol{\nabla} p\|_{L^2(\Omega)}.$$

Thus

$$\sup_{\boldsymbol{v} \in L^2(\Omega)} \frac{\displaystyle\int_{\Omega} \boldsymbol{v} \cdot \boldsymbol{\nabla} p}{\|\boldsymbol{v}\|_{L^2(\Omega)}} \geq \|\boldsymbol{\nabla} p\|_{L^2(\Omega)}.$$

This shows that the inf-sup condition is satisfied and the problem is well-posed.

9) Assume that $d = 3$ and let $\mathcal{T}_h$ be a mesh of tetrahedra. Set

$$X_h = \{\boldsymbol{v}_h \in (L^2(\Omega))^d, \forall K \in \mathcal{T}_h, \boldsymbol{v}_h|_K \in (\mathbb{P}_0(K))^d\},$$

$$M_h = \{q_h \in C^0(\Omega), \forall K \in \mathcal{T}_h, q_h|_K \in \mathbb{P}_1(K)\} \cap H^1_{\int=0}(\Omega).$$

Show that the discrete problem resulting from the discretization of problem (Q) in X_h and M_h is well-posed.

$a(\boldsymbol{u}, \boldsymbol{v})$ is coercive on X_h.

Moreover, if $p_h \in M_h$, then $\boldsymbol{\nabla} p_h$ is a piecewise constant function. Thus for $\boldsymbol{v}_h = \boldsymbol{\nabla} p_h \in X_h$.

$$\frac{\int_\Omega \boldsymbol{\nabla} p_h \cdot \boldsymbol{v}_h}{\|\boldsymbol{v}_h\|_{L^2(\Omega)}} = \|\boldsymbol{\nabla} p_h\|_{L^2(\Omega)}.$$

The inf-sup condition is satisfied and the problem is well-posed.