CME358 - Assignment 3

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Let Ω be a smooth domain of $\mathbb{R}^d$, and $\boldsymbol{n}$ the outward unit normal on $\partial\Omega$. Let $\boldsymbol{f} \in (L^2(\Omega))^d$, $g \in H^{1/2}(\partial\Omega)$ and $k \in L^2(\Omega)$. This problem deals with two different weak formulations of the Darcy equations.

Part 1

Consider the problem: Search for $(\boldsymbol{u}, p) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ such that for all $(\boldsymbol{v}, q) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$

$$(P) \left\{ \begin{array}{rcl} \displaystyle\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} - \int_\Omega p \,\mathrm{div}\, \boldsymbol{v} &=& \displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\partial\Omega} g\boldsymbol{v}\cdot\boldsymbol{n} \\ \displaystyle\int_\Omega q \,\mathrm{div}\, \boldsymbol{u} &=& \displaystyle\int_\Omega kq \end{array} \right.$$

1) What are the partial differential equations and the boundary conditions corresponding to this problem? Are the boundary conditions enforced naturally or essentially ?

2) Let $q \in L^2(\Omega)$. Prove there exists a unique $\Phi \in H^1_0(\Omega)$ such that

$$\int_\Omega \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\psi = \int_\Omega q\psi, \qquad \text{for all } \psi \in H^1_0(\Omega).$$

Set $\boldsymbol{v} = \boldsymbol{\nabla}\Phi$. Give an estimate (upper bound) of $\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)}$.

3) Deduce from the previous question that there exists $\beta > 0$ such that for all $q \in L^2(\Omega)$,

$$\sup_{\boldsymbol{w}\in H(\mathrm{div};\Omega)} \frac{-\displaystyle\int_\Omega q \,\mathrm{div}\, \boldsymbol{w}}{\|\boldsymbol{w}\|_{H(\mathrm{div};\Omega)}} \geq \beta \|q\|_{L^2(\Omega)}. \tag{1}$$

4) Let $q \in L^2(\Omega)$. Set $\alpha = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega q$. Construct $\boldsymbol{v}_\alpha \in (H^1(\Omega))^d$ such that $\mathrm{div}\, \boldsymbol{v}_\alpha = \alpha$. Prove that the operator div is surjective from $(H^1(\Omega))^d$ onto $L^2(\Omega)$.

Hint: Use the following result from the course: the operator div *is surjective from* $(H^1_0(\Omega))^d$ *onto* $L^2_0(\Omega)$.

5) Deduce from the previous question another proof of (1).

6) Prove that problem (P) is well-posed.

Part 2

7) Assume that $\int_\Omega k + \int_{\partial\Omega} g = 0$ and define the space $H^1_{\int=0}(\Omega) = \{q \in H^1(\Omega), \int_\Omega q = 0\}$ equipped with the norm $\|\boldsymbol{\nabla}\cdot\|_{L^2(\Omega)}$. Consider the problem: search for $(\boldsymbol{u}, p) \in (L^2(\Omega))^d \times H^1_{\int=0}(\Omega)$ such that for all $(\boldsymbol{v}, q) \in (L^2(\Omega))^d \times H^1_{\int=0}(\Omega)$

$$(Q) \left\{ \begin{array}{rcl} \displaystyle\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{\nabla}p\cdot\boldsymbol{v} &=& \displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \\ \displaystyle -\int_\Omega \boldsymbol{\nabla}q\cdot\boldsymbol{u} &=& \displaystyle\int_\Omega kq + \int_{\partial\Omega} gq \end{array} \right.$$

What are the partial differential equations and the boundary conditions corresponding to this problem? Are the boundary conditions enforced naturally or essentially ?

Hint: Prove that $\boldsymbol{u} \in H(div;\Omega)$ *and admit the fact that every function in* $H(div,\Omega)$ *has a well-defined normal trace on* $\partial\Omega$

8) Prove that problem (Q) is well-posed.

9) Assume that $d = 3$ and let $\mathcal{T}_h$ be a mesh of tetrahedra. Set

$$X_h = \{\boldsymbol{v}_h \in (L^2(\Omega))^d, \forall K \in \mathcal{T}_h, \boldsymbol{v}_h|_K \in (\mathbb{P}_0(K))^d\},$$

$$M_h = \{q_h \in C^0(\Omega), \forall K \in \mathcal{T}_h, q_h|_K \in \mathbb{P}_1(K)\} \cap H^1_{\int=0}(\Omega).$$

Show that the discrete problem resulting from the discretization of problem (Q) in X_h and M_h is well-posed.