CME358 - Assignment 2
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Exercise 1 (Penalization method) Let $(X, \|\cdot\|_X)$ and $(M, \|\cdot\|_M)$ be two Hilbert spaces. Denote by $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_M$ the scalar products associated to the norms $\|\cdot\|_X$ and $\|\cdot\|_M$. For $f \in X'$ and $g \in M'$, consider the following problem: search for $(u,p) \in X \times M$ such that for all $(v,q) \in X \times M$:

$$(P)\left\{\begin{array}{rcl} a(u,v) + b(v,p) &=& < f, v >, \\ b(u,q) &=& < g, q > . \end{array}\right.$$

Assume that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bilinear continuous forms on $X \times X$ and $X \times M$ respectively. Assume there exists $\beta > 0$ such that:

$$\forall q \in M, \exists\, v \in X, v \neq 0, b(v,q) \geq \beta \|v\|_X \|q\|_M. \tag{1}$$

and assume that $a(\cdot,\cdot)$ is $\alpha-$coercive.

Let $c(\cdot,\cdot)$ be a bilinear continuous and $\gamma-$coercive form on $M \times M$ and let $C \in \mathcal{L}(M, M')$ be defined by

$$< Cp, q >= c(p,q), \qquad \forall p, q \in M.$$

Operator A and B are defined accordingly, associated to $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ respectively.

1) Prove that problem (P) is well-posed.

For $0 < \varepsilon < 1$, consider the following problem: search for $(u_\varepsilon, p_\varepsilon) \in X \times M$ such that for all $(v,q) \in X \times M$,

$$(P_\varepsilon)\left\{\begin{array}{rcl} a(u_\varepsilon,v) + b(v,p_\varepsilon) &=& < f, v >, \\ -\varepsilon c(p_\varepsilon,q) + b(u_\varepsilon,q) &=& < g, q > . \end{array}\right.$$

2) Prove that (P_ε) is equivalent to finding $(u_\varepsilon, p_\varepsilon) \in X \times M$ such that

$$a(u_\varepsilon,v) + \frac{1}{\varepsilon} < C^{-1}Bu_\varepsilon, Bv > \;=\; < f, v > + \frac{1}{\varepsilon} < C^{-1}g, Bv >, \forall v \in X \tag{2}$$

$$p_\varepsilon \;=\; \frac{1}{\varepsilon} C^{-1}(Bu_\varepsilon - g). \tag{3}$$

3) Prove that the problem of the previous question has a unique solution. Is condition (1) necessary to prove that result?

4) Prove the following inequalities:

$$\|p_\varepsilon - p\|_M \leq \frac{\|a\|}{\beta} \|u_\varepsilon - u\|_X, \tag{4}$$

$$a(u_\varepsilon - u, u_\varepsilon - u) \leq \varepsilon \frac{\|a\|\,\|c\|}{\beta} \|u_\varepsilon - u\|_X \|p\|_M, \tag{5}$$

5) Conclude that there exists $C_2 > 0$ independent of ε such that

$$\|u - u_\varepsilon\|_X + \|p - p_\varepsilon\|_M \leq C_2 \varepsilon \left(\|f\|_{X'} + \|g\|_{M'}\right). \tag{6}$$

6) Let $\varepsilon > 0$. Consider the problem: find $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in (H^1_0(\Omega))^3 \times L^2_0(\Omega)$ such that:

$$\left\{\begin{array}{rcl} -\nu \Delta \boldsymbol{u}_\varepsilon + \boldsymbol{\nabla} p_\varepsilon &=& f, \\ p_\varepsilon &=& -\dfrac{1}{\varepsilon} \operatorname{div} \boldsymbol{u}_\varepsilon. \end{array}\right. \tag{7}$$

Prove that this problem is well-posed and that, when ε goes to 0, its solution goes to the solution of a problem to be determined.

7) Prove that (7) can be written as a minimization problem. Discuss the advantages and the drawbacks of the resolution of this problem.

Exercise 2 (Augmented Lagrangian method) Let $M \in \mathbb{N}$ and $N \in \mathbb{N}$ with $M < N$. Denote by $|\cdot|$ the Euclidian norm in $\mathbb{R}^M$ or $\mathbb{R}^N$ and $(\cdot, \cdot)$ the associated scalar product. Let A be a $N \times N$ symmetric positive definite matrix and $b \in \mathbb{R}^N$. Let B be a full-rank matrix $M \times N$. Define $\mathcal{L}$ from $\mathbb{R}^N \times \mathbb{R}^M$ on $\mathbb{R}$ by:

$$\mathcal{L}(v, q) = J(v) + (q, Bv)$$

with $J(v) = \frac{1}{2}(Av, v) - (b, v)$. Let us denote by (u, p) a saddle-point of $\mathcal{L}$: for all $(v, q) \in \mathbb{R}^N \times \mathbb{R}^M$: $\mathcal{L}(u, q) \le \mathcal{L}(u, p) \le \mathcal{L}(v, p)$. Let $r > 0$, let us define

$$\mathcal{L}_r(v, q) = \mathcal{L}(v, q) + \frac{r}{2}|Bv|^2.$$

Define the sequences $(u_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ as follows: let $p_0 \in \mathbb{R}^M$, for $n \ge 0$, assuming p_n is known, compute $u_n \in \mathbb{R}^N$ solution to

$$\mathcal{L}_r(u_n, p_n) \le \mathcal{L}_r(v, p_n), \forall v \in \mathbb{R}^N, \tag{8}$$

then, set

$$p_{n+1} = p_n + \rho_n B u_n, \tag{9}$$

where ρ_n is a given positive number. Assume that $\forall n, 0 < \alpha \le \rho_n \le 2r$, where α is given. Set $\delta u_n = u - u_n$ and $\delta p_n = p - p_n$.

1) Show that (8) is equivalent to

$$(A + rB^T B)u_n + B^T p_n = b$$

2) Show that

$$|\delta p_n|^2 - |\delta p_{n+1}|^2 = 2\rho_n(A\delta u_n, \delta u_n) + \rho_n(2r - \rho_n)|B\delta u_n|^2$$

3) Show that δp_n converges. Deduce that $u_n \to u$ and $p_n \to p$ as $n \to \infty$.

4) Show that any saddle point of $\mathcal{L}$ is a saddle point of $\mathcal{L}_r$ and conversely.