Exercise 1 (Penalization method) Let $(X, \|\cdot\|_X)$ and $(M, \|\cdot\|_M)$ two Hilbert spaces. We denote by $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_M$ the scalar products associated to the norms $\|\cdot\|_X$ and $\|\cdot\|_M$. For $f \in X'$ and $g \in M'$, we are interested in the solution of the following problem: search for $(u,p) \in X \times M$ such that for all $(v,q) \in X \times M$:

$$(P) \left\{ \begin{array}{rcl} a(u,v) + b(v,p) & = & < f, v >, \\ b(u,q) & = & < g, q > . \end{array} \right.$$

We assume that $a(\cdot,\cdot)$ et $b(\cdot,\cdot)$ are bilinear continuous forms on $X \times X$ and $X \times M$ respectively. We assume there exists $\beta > 0$ such that:

$$\forall q \in M, \exists\, v \in X, v \neq 0, b(v,q) \geq \beta \|v\|_X \|q\|_M. \tag{1}$$

and that $a(\cdot,\cdot)$ is $\alpha-$coercive.

Let $c(\cdot,\cdot)$ be a bilinear continuous and $\gamma-$coercive form on $M \times M$ and let $C \in \mathcal{L}(M, M')$ be defined by

$$< Cp, q >= c(p,q), \qquad \forall p, q \in M.$$

We define analogously operator A and B associated to $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ respectively.

1) Prove that problem (P) is well-posed.

Let $V = \text{Ker } B$. a is coercive on $X \times X$, thus also on $V \times V$. Moreover, b satisfies the inf-sup condition because of Eq. (1). Therefore Problem (P) is well-posed.

For $0 < \varepsilon < 1$, we consider the following problem: find $(u_\varepsilon, p_\varepsilon) \in X \times M$ such that for all $(v,q) \in X \times M$,

$$(P_\varepsilon) \left\{ \begin{array}{rcl} a(u_\varepsilon,v) + b(v,p_\varepsilon) & = & < f, v >, \\ -\varepsilon c(p_\varepsilon, q) + b(u_\varepsilon,q) & = & < g, q > . \end{array} \right.$$

2) Prove that (P_ε) is equivalent to finding $(u_\varepsilon, p_\varepsilon) \in X \times M$ such that

$$a(u_\varepsilon, v) + \frac{1}{\varepsilon} < C^{-1}Bu_\varepsilon, Bv > \; = \; < f, v > + \frac{1}{\varepsilon} < C^{-1}g, Bv >, \forall v \in X \tag{2}$$

$$p_\varepsilon \; = \; \frac{1}{\varepsilon} C^{-1}(Bu_\varepsilon - g). \tag{3}$$

$\forall q \in M$,

$$
\begin{aligned}
-\epsilon c(p_\epsilon, q) &= < g, q > - b(u_\epsilon, q) \\
-C p_\epsilon &= \frac{1}{\epsilon} g - \frac{1}{\epsilon} B u_\epsilon \\
p_\epsilon &= -\frac{1}{\epsilon} C^{-1} g + \frac{1}{\epsilon} C^{-1} B u_\epsilon.
\end{aligned}
$$

C is invertible since c is coercive on $M \times M$ with constant $\gamma > 0$. Hence

$$
\begin{aligned}
b(v, p_\epsilon) &= < B^T p_\epsilon, v > \\
&= \frac{1}{\epsilon} < B^T C^{-1} B u_\epsilon, v > - \frac{1}{\epsilon} < B^T C^{-1} g, v > \\
&= \frac{1}{\epsilon} < C^{-1} B u_\epsilon, B v > - \frac{1}{\epsilon} < C^{-1} g, B v >
\end{aligned}
$$

3) Prove that the problem of the previous question has a unique solution. Is condition (1) necessary to prove that result?

Defining $a_\epsilon(u, v) = a(u, v) + \frac{1}{\epsilon} < C^{-1} B u, B v >$, a_ϵ is coercive:

$$
\begin{aligned}
a_\epsilon(u, u) &= a(u, u) + \frac{1}{\epsilon} < C^{-1} B u, B u > \\
&= a(u, u) + \frac{1}{\epsilon} < q, C q > \\
&\geq \alpha \|u\|_X^2 + \frac{\gamma}{\epsilon} \|q\|_M^2 \\
&\geq \alpha \|u\|_X^2,
\end{aligned}
$$

where q is defined by $Cq = Bu$.

Moreover, a_ϵ is continuous since,

$$
\begin{aligned}
a_\epsilon(u, v) &= a(u, v) + \frac{1}{\epsilon} < q, B v > \\
&\leq \|a\| \|u\|_X \|v\|_X + \frac{1}{\epsilon} \|B\| \|q\|_M \|v\|_X \\
&\leq \|a\| \|u\|_X \|v\|_X + \frac{\|B\|^2}{\epsilon \gamma} \|u\|_X \|v\|_X
\end{aligned}
$$

where the following inequality has been used

$$
\gamma \|q\|_M^2 \leq < C q, q > = < B u, q > \leq \|B\| \|q\|_M \|u\|_X.
$$

Moreover, the right hand side belongs to X'. From the Lax-Milgram theorem, this problem is well-posed. Note that the inf-sup condition is not necessary here.

4) Prove the following inequalities:

$$\|p_\varepsilon - p\|_M \leq \frac{\|a\|}{\beta}\|u_\varepsilon - u\|_X, \tag{4}$$

$$a(u_\varepsilon - u, u_\varepsilon - u) \leq \varepsilon\frac{\|a\|\,\|c\|}{\beta}\|u_\varepsilon - u\|_X\|p\|_M, \tag{5}$$

Starting from

$$\left\{\begin{array}{rcl} Au_\varepsilon + B^T p_\varepsilon & = & f, \\ Bu_\varepsilon - \varepsilon C p_\varepsilon & = & g \end{array}\right. \tag{6}$$

and

$$\left\{\begin{array}{rcl} Au + B^T p & = & f, \\ Bu & = & g, \end{array}\right. \tag{7}$$

$A(u - u_\varepsilon) + B^T(p - p_\varepsilon) = 0$. using the inf-sup condition,

$$\|B^T(p - p_\varepsilon)\|_{X'} \geq \beta\|p - p_\varepsilon\|_M.$$

As

$$\|A(u - u_\varepsilon)\|_{X'} \leq \|A\|\|u - u_\varepsilon\|_X = \|a\|\|u - u_\varepsilon\|_X.$$

$$\|p - p_\varepsilon\|_M \leq \frac{\|a\|}{\beta}\|u - u_\varepsilon\|_X.$$

Since $A(u_\varepsilon - u) + B^T(p_\varepsilon - p) = 0$,

$$\begin{aligned} < A(u_\varepsilon - u), u_\varepsilon - u > &= < p - p_\varepsilon, B(u - u_\varepsilon) > \\ &= < p - p_\varepsilon, \varepsilon C p_\varepsilon > \\ &= < p - p_\varepsilon, \varepsilon C(p_\varepsilon - p_) > + < p - p_\varepsilon, \varepsilon C p > \end{aligned}$$

since $< p - p_\varepsilon, \varepsilon C(p_\varepsilon - p) > \leq 0$,

$$\begin{aligned} < A(u_\varepsilon - u), u_\varepsilon - u > &\leq < p - p_\varepsilon, \varepsilon C p > \\ &\leq \varepsilon\|c\|\|p - p_\varepsilon\|_M\|p\|_M \\ &\leq \frac{\varepsilon\|a\|\|c\|}{\beta}\|u_\varepsilon - u\|_X\|p\|_M \end{aligned}$$

Thus,

$$\|u - u_\varepsilon\|_X \leq \frac{\varepsilon\|a\|\|c\|}{\alpha\beta}\|p\|_M.$$

5) Conclude that there exists $C_2 > 0$ independent of ε such that

$$\|u - u_\varepsilon\|_X + \|p - p_\varepsilon\|_M \leq C_2\varepsilon\left(\|f\|_{X'} + \|g\|_{M'}\right). \tag{8}$$

We know that

$$\|p\|_M \leq \frac{\|a\|}{\beta^2}\left(1+\frac{\|a\|}{\alpha}\right)\|g\|_{M'} + \frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right)\|f\|_{X'}.$$

Thus

$$\|u-u_\varepsilon\|_X \leq C_1\varepsilon(\|f\|_{X'}+\|g\|_{M'})$$

and

$$\|p-p_\varepsilon\|_M \leq \frac{\|a\|}{\beta}C_1\varepsilon(\|f\|_{X'}+\|g\|_{M'}),$$

therefore,

$$\|u-u_\varepsilon\|_X + \|p-p_\varepsilon\|_M \leq C_2\varepsilon\left(\|f\|_{X'}+\|g\|_{M'}\right).$$

6) Let $\varepsilon > 0$. Consider the problem: find $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in (H_0^1(\Omega))^3 \times L_0^2(\Omega)$ such that:

$$\begin{cases} -\nu\Delta\boldsymbol{u}_\varepsilon + \boldsymbol{\nabla} p_\varepsilon &= f, \\ p_\varepsilon &= -\dfrac{1}{\varepsilon}\operatorname{div}\boldsymbol{u}_\varepsilon. \end{cases} \tag{9}$$

Prove that this problem is well-posed and that, when ε goes to 0, its solution goes to the solution of a problem to be determined.

Applying the result from question 1 with $A = -\nu\Delta$, $B = -\operatorname{div}$, $C = Id$ and $g = 0$, the problem is well-posed.

According to question 5, $u_\varepsilon \xrightarrow{H_0^1} u$ and $p_\varepsilon \xrightarrow{L^2} p$ where (u,p) is solution to the Stokes problem.

7) If $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are symmetric, prove that (9) can be written as a minimization problem. Discuss the advantages and the drawbacks of the resolution of this problem.

From

$$-\nu\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla\operatorname{div} u_\varepsilon = f$$

one can define

$$J(u) = \frac{\nu}{2}\int|\nabla u|^2 + \frac{1}{2\varepsilon}\int|\operatorname{div} u|^2 - \int fu.$$

From the Lax-Milgram theorem,

$$J(u_\varepsilon) = \inf J(u).$$

Resolution of Eq. (9) does not need the inf-sup condition. (On the other hand, convergence of u_ε to u and p_ε to p need the inf-sup condition). The presence of $\frac{1}{\varepsilon}$ deteriorates the condition number of the problem and may result in the locking phenomena.

Exercise 2 (Augmented Lagrangian method) Let $M \in \mathbb{N}$ and $N \in \mathbb{N}$ with $M < N$. We denote by $|\cdot|$ the Euclidian norm in $\mathbb{R}^M$ or $\mathbb{R}^N$ and $(\cdot,\cdot)$ the associated scalar product. Let A be a $N \times N$ symmetric positive definite matrix and $b \in \mathbb{R}^N$. Let B a full-rank matrix $M \times N$. Define $\mathcal{L}$ from $\mathbb{R}^N \times \mathbb{R}^M$ on $\mathbb{R}$ by:

$$\mathcal{L}(v,q) = J(v) + (q, Bv)$$

with $J(v) = \frac{1}{2}(Av,v) - (b,v)$. Lets denote by (u,p) a saddle-point of $\mathcal{L}$: for all $(v,q) \in \mathbb{R}^N \times \mathbb{R}^M$: $\mathcal{L}(u,q) \leq \mathcal{L}(u,p) \leq \mathcal{L}(v,p)$. Let $r > 0$, lets define

$$\mathcal{L}_r(v,q) = \mathcal{L}(v,q) + \frac{r}{2}|Bv|^2.$$

We define the sequences $(u_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ as follows. Let $p_0 \in \mathbb{R}^M$. For $n \geq 0$, assuming p_n is known, we compute $u_n \in \mathbb{R}^N$ solution to

$$\mathcal{L}_r(u_n,p_n) \leq \mathcal{L}_r(v,p_n), \forall v \in \mathbb{R}^N, \tag{10}$$

then, we set

$$p_{n+1} = p_n + \rho_n B u_n, \tag{11}$$

where ρ_n is a given positive number. We assume that $\forall n, 0 < \alpha \leq \rho_n \leq 2r$, where α is given. We set $\delta u_n = u - u_n$ and $\delta p_n = p - p_n$.

1) Show that (10) **is equivalent to**

$$(A + rB^T B)u_n + B^T p_n = b$$

$$\mathcal{L}_r(v,p_n) = \mathcal{L}(v,p_n) + \frac{r}{2}|Bv|^2 = \frac{1}{2}(Av,v) - (b,v) + (p_n,Bv) + \frac{r}{2}|Bv|^2.$$

Let $\epsilon > 0$.

$$\mathcal{L}_r(v+\epsilon w,p_n) - \mathcal{L}_r(v,p_n) = \epsilon\left(\frac{1}{2}(Aw,v) + \frac{1}{2}(Av,w) - (b,w) + (p_n,Bw) + r(B^TBv,w)\right).$$

Since A is symmetric

$$\mathcal{L}_r(v+\epsilon w,p_n) - \mathcal{L}_r(v,p_n) = \epsilon\left((Av,w) - (b,w) + (B^Tp_n,w) + r(B^TBv,w)\right).$$

Thus

$$\mathcal{L}'_r(v,p_n).w = (Av,w) - (b,w) + (B^Tp_n,w) + r(B^TBv,w)$$

and if u_n is a global minimum in the open set $\mathbb{R}^N$, necessarily

$$\mathcal{L}_r'(u_n, p_n).w = 0, \quad \forall w \in \mathbb{R}^N \Leftrightarrow Au_n - b + B^T p_n + rB^T Bu_n.$$

2) Show that

$$|\delta p_n|^2 - |\delta p_{n+1}|^2 = 2\rho_n(A\delta u_n, \delta u_n) + \rho_n(2r - \rho_n)|B\delta u_n|^2$$

$$\begin{aligned} |\delta p_n|^2 - |\delta p_{n+1}|^2 &= |\delta p_n|^2 - |\delta p_n - \rho_n Bu_n|^2 \\ &= 2(\rho_n Bu_n, \delta p_n) - \rho_n^2 |Bu_n|^2 \end{aligned}$$

Since $Bu = 0$,

$$|\delta p_n|^2 - |\delta p_{n+1}|^2 = -2(\rho_n B\delta u_n, \delta p_n) - \rho_n^2 |Bu_n|^2$$

Moreover,

$$Au + B^T p = b$$

and

$$Au_n + B^T p_n = b - rB^T Bu_n.$$

Therefore,

$$B^T \delta p_n = -A\delta u_n + rB^T Bu_n$$

and

$$\begin{aligned} |\delta p_n|^2 - |\delta p_{n+1}|^2 &= -2(\rho_n \delta u_n, -A\delta u_n + rB^T Bu_n) - \rho_n^2 |Bu_n|^2 \\ &= 2\rho_n(A\delta u_n, \delta u_n) - 2\rho_n r(\delta u_n, B^T(-B\delta u_n)) - \rho_n |B\delta u_n|^2 \\ &= 2\rho_n(A\delta u_n, \delta u_n) + \rho_n(2r - \rho_n)|B\delta u_n|^2 \end{aligned}$$

3) Show that δp_n converges. Deduce that $u_n \to u$ and $p_n \to p$ as $n \to \infty$.

Since $2r - \rho_n > 0$, and A is symmetric positive definite, $|\delta p_n|^2 - |\delta p_{n+1}|^2 \leq 0$ and thus the sequence $|\delta p_n|^2$ is decreasing and lower bounded by zero. Therefore it converges.

Since

$$(A + rB^T B)\delta u_n = -B^T \delta p_n$$

and the matrix $(A + rB^T B)$ is symmetric definite positive (SPD), thus invertible, δu_n converges as well.

Since $2\rho_n(A\delta u_n, \delta u_n) + \rho_n(2r - \rho_n)|B\delta u_n|^2$ converges to zero and is the sum of two positive terms, each term converges to zero. In particular $|2\rho_n(A\delta u_n, \delta u_n)| \geq |2\alpha(A\delta u_n, \delta u_n)| > 0$ converges to zero and so does δu_n since A is SPD. Thus $u_n \to u$.

Going back to

$$B^T \delta p_n = -(A + rB^T B)\delta u_n$$

since B is of full-rank M, $B^T \in \mathbb{R}^{N \times M}$ is injective and $\delta p_n \to 0$, thus $p_n \to p$.

4) Show that any saddle point of $\mathcal{L}$ is a saddle point of $\mathcal{L}_r$ and conversely.

- Let (u, p) be a saddle point of $\mathcal{L}$. Thus,

  $$Au + B^T p = b, \quad Bu = 0.$$

  Letting $(v, q) \in \mathbb{R}^N \times \mathbb{R}^M$,

  $$\mathcal{L}_r(u, q) = \mathcal{L}(u, q) \leq \mathcal{L}(u, p) = \mathcal{L}_r(u, p) \leq \mathcal{L}(v, p) \leq \mathcal{L}_r(v, p).$$

  (u, p) is a saddle point of $\mathcal{L}_r$.

- Let (u_r, p_r) be a saddle point of $\mathcal{L}_r$. We showed that

  $$(A + rB^T B)u_r = b - B^T p_r.$$

  Since $\mathcal{L}_r(u_r, q) \leq \mathcal{L}_r(u_r, p_r), \quad \forall q \in \mathbb{R}^M$,

  $$(q, Bu_r) \leq (p_r, Bu_r), \quad \forall q \in \mathbb{R}^M.$$

  Necessarily, choosing $q \neq 0$ and $-q$ shows that $Bu_r = 0$.

  Then $Au_r = b - B^T p_r$. These two equalities show that (u_r, p_r) is a saddle point for $\mathcal{L}$ as well.