Consider the following problem:

$$
\begin{array}{rcll}
-\nu u'' + \beta u' & = & 1, & \text{in } \Omega = (0,1), \\
u(0) = u(1) & = & 0,
\end{array}
\tag{1}
$$

where ν and β are two positive constants.

1) Give a physical interpretation of the different terms in (1).

2) Compute the exact solution to (1).

3) Plot the exact solution for $\beta = 1$, $\nu = 1, 0.1$ and 0.01. Note the *boundary layer* in $x = 1$. Propose an estimation of the thickness of this boundary layer as a function of ν and β when β/ν is large.

4) Propose a variational formulation of the equation and show that the problem is well-posed in $H_0^1(\Omega)$.

We wish to approximate this problem with the $\mathbb{P}_1$ finite element on a uniform mesh with step $h = 1/(N+1)$. We denote by $(\phi_1, \ldots, \phi_N)$ the classical hat function basis of $V_h = X_h^1 \cap H_0^1(\Omega)$. The approximate solution u_h is solution to the problem: find $u_h \in V_h$ such that for all $v_h \in V_h$

$$
a(u_h, v_h) = \int_0^1 v_h
\tag{2}
$$

5) Introduce the Peclet number:

$$
\gamma = \frac{h\beta}{\nu}
$$

and compute the matrix associated to problem (2).

6) Simulations in Matlab. Take $\beta = 1$, $h = 0.01$ and plot the solutions to (2) for values of ν corresponding to $\gamma = 1$ and $\gamma = 5$. Try different values of γ to determine an instability threshold. What is the meaning of this threshold for the matrix of the problem?

7) Let $b : [0,1] \to \mathbb{R}$ be a smooth function such that $b(0) = b(1) = 0$ and $b(\xi) > 0$ on $(0,1)$ (b is called a *bubble function*). We propose to replace the test function space V_h by another space W_h, while keeping the same space V_h to search for the solution. We define W_h spanned by the functions ψ defined by:

$$
\psi_i = \phi_i +
\begin{cases}
b\left(\dfrac{x - x_{i-1}}{h}\right), & x \in [x_{i-1}, x_i], \\[2ex]
-b\left(\dfrac{x - x_i}{h}\right), & x \in [x_i, x_{i+1}],
\end{cases}
$$

where $x_i = ih$ is the ith node of the mesh. Choose an arbitrary bubble function b and plot the basis function ψ_i. Comment.

8) Consider the problem: search for $\bar{u}_h \in V_h$ such that

$$
a(\bar{u}_h, v_h) = \int_0^1 v_h, \qquad \forall v_h \in W_h
\tag{3}
$$

Compute the matrix associated to this problem. **Remark:** The approach consisting of choosing different spaces for the solution and the test functions is referred to as the *Petrov-Galerkin method.*

9) Show that the matrix resulting from the Petrov-Galerkin method (3) is the same as that resulting from the Galerkin method (2) up to the modification of the diffusion coefficient. Show that problem (3) is

equivalent to searching for $\bar{u}_h \in V_h$ such that

$$a_h(\bar{u}_h, v_h) = \int_0^1 v_h, \qquad \forall v_h \in V_h \tag{4}$$

where a_h is a bilinear form which depends on h.

10) How to choose $\int_0^1 b$ as a function of γ such that problem (4) is always stable?

11) Simulation in Matlab. Take $\beta = 1$, $h = 0.01$, a bubble function of your choice. Plot the solution to (4) for $\gamma = 1$ and $\gamma = 5$ (depending on your choice for b, you can choose other values if these ones do not satisfy the criterion of the previous question). Comment.

12) Prove and comment the following error estimate:

$$|u - \bar{u}_h|_1 \leq \inf_{w_h \in V_h} \left(\left(1 + \frac{\|a\|}{\nu}\right) |u - w_h|_1 + \frac{1}{\nu} \sup_{v_h \in V_h} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{|v_h|_1} \right)$$

Hint: First notice that, generally speaking, if $a(\cdot, \cdot)$ is α-coercive, one always have:

$$\alpha \|u\|_X \leq \sup_{v \in X} \frac{a(u, v)}{\|v\|_X}$$

Next prove:

$$\nu |u_h - w_h|_1 \leq \sup_{v_h \in V_h} \frac{a_h(u_h - w_h, v_h)}{|v_h|_1}$$

and

$$a_h(\bar{u}_h - w_h, v_h) = a(u - w_h, v_h) + a(w_h, v_h) - a_h(w_h, v_h)$$