Exercise 1 (1D Petrov-Galerkin for advection-diffusion) Consider the following problem:

$$\begin{array}{rcll} -\nu u'' + \beta u' & = & 1, & \text{in } \Omega = (0,1), \\ u(0) = u(1) & = & 0, & \end{array} \quad (1)$$

where ν and β are two positive constants.

1) Give a physical interpretation of the different terms in (1).
$-\nu u''$ is a diffusion term, $\beta u'$ an advection term and the right-hand side 1 is a source term.

2) Compute the exact solution to (1).

Integrating the ODE once leads to

$$-\nu u' + \beta u = x + C$$

with C being a constant to be determined by the boundary conditions. The associated homogeneous differential equation $\nu u' + \beta u = 0$ can be integrated as:

$$u(x) = A \exp\left(\frac{\beta x}{\nu}\right)$$

Using the method of variation of the constant, we are looking for a solution of the non-homogeneous ODE under the form:

$$u(x) = A(x) \exp\left(\frac{\beta x}{\nu}\right)$$

One finds

$$A(x) = \frac{1}{\beta} \exp\left(-\frac{\beta x}{\nu}\right) \left(x + \frac{\nu}{\beta} + C\right) + D$$

where D is as constant. The two boundary conditions give the values of the constants:

$$C = -\frac{1}{\exp\left(\frac{\beta}{\nu}\right) - 1} - \frac{\nu}{\beta}$$

and

$$D = -\frac{1}{\beta\left(\exp\left(\frac{\beta}{\nu}\right) - 1\right)}$$

Therefore, the exact solution to (1) is:

$$u(x) = \frac{1}{\beta}\left(x - \frac{\exp\left(\frac{\beta x}{\nu}\right) - 1}{\exp\left(\frac{\beta}{\nu}\right) - 1}\right) \quad (2)$$

3) Plot the exact solution for $\beta = 1$, $\nu = 1, 0.1$ and 0.01. Note the *boundary layer* in $x = 1$. Propose an estimation of the thickness of this boundary layer as a function of ν and β when β/ν is large.
The exact solution is reported in Figure 1.

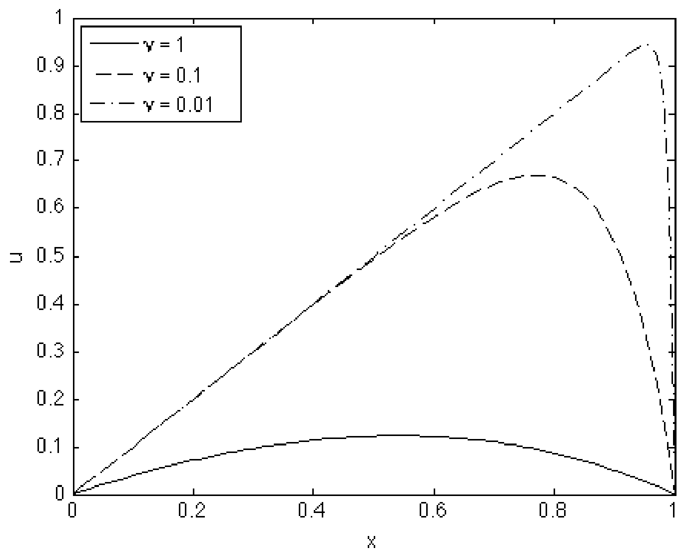

Figure 1: Exact solution of the ODE for various values of ν

The thickness of the boundary layer $\delta = 1 - x^\star$ verifies $u'(x^\star) = 0$. Since

$$u'(x^\star) = \frac{1}{\beta}\left(1 - \frac{\beta}{\nu}\frac{\exp\left(\frac{\beta x^\star}{\nu}\right)}{\exp\left(\frac{\beta}{\nu}\right) - 1}\right)$$

$x^\star$ verifies:

$$x^\star = \frac{\nu}{\beta}\left(\ln\left(\frac{\nu}{\beta}\right) + \ln\left(\exp\left(\frac{\beta}{\nu}\right) - 1\right)\right)$$

when β/ν is large,

$$x^\star \sim \frac{\nu}{\beta}\ln\left(\frac{\nu}{\beta}\right) + 1$$

and

$$\delta \sim -\frac{\nu}{\beta}\ln\left(\frac{\nu}{\beta}\right).$$

The larger ν, the thicker the boundary layer.

4) Propose a variational formulation of the equation and show that the problem is well-posed in $H_0^1(\Omega)$.

Considering the Hilbert Space $\left(H_0^1(\Omega), \|\cdot\|_1\right)$, a variational formulation of the equation can be written as:

$$\nu \int_0^1 u'v'dx + \beta \int_0^1 u'vdx = \int_0^1 vdx, \quad \forall v \in H_0^1(\Omega) \tag{3}$$

where $u \in H_0^1(\Omega)$ is the exact solution to the problem.
Let $a(u,v)$ denote the bilinear form:

$$a(u,v) = \nu \int_0^1 u'v'dx + \beta \int_0^1 u'vdx$$

a is continuous since $\forall u, v \in H_0^1(\Omega)$:

$$|a(u,v)| \leq \nu \left(\left| \int_0^1 u' \left(v' + \frac{\beta}{\nu} v \right) dx \right| \right)$$

Using Cauchy-Schwartz inequality,

$$\begin{aligned}
|a(u,v)| &\leq \nu \sqrt{\int_0^1 u'^2 dx} \sqrt{\int_0^1 \left(v' + \frac{\beta}{\nu} v \right)^2 dx} \\
&\leq \nu \|u\|_1 \sqrt{\int_0^1 \left(v'^2 + \frac{\beta^2}{\nu^2} v^2 \right) dx} \\
&\leq \nu \max\left(1, \frac{\beta}{\nu}\right) \|u\|_1 \|v\|_1 \\
&\leq \max(\nu, \beta) \|u\|_1 \|v\|_1
\end{aligned}$$

using the property $\int_0^1 vv'dx = 0$. This shows the continuity.
Let $v \in H_0^1(\Omega)$:

$$\begin{aligned}
a(v,v) &= \nu \int_0^1 v'^2 dx + \beta \int_0^1 vv'dx \\
&= \nu \int_0^1 v'^2 dx + \frac{\beta}{2} [v^2]_0^1 \\
&= \nu \int_0^1 v'^2 dx \\
&\geq \frac{\nu}{2} \left(\int_0^1 v'^2 + \frac{1}{C_\Omega} \int_0^1 v^2 dx \right)
\end{aligned}$$

using Poincare's inequality. Therefore:

$$a(v,v) \geq \min\left(\frac{\nu}{2}, \frac{\nu}{2C_\Omega}\right) \|v\|_1^2$$

which shows the coercivity of a.
The conditions of Lax-Milgram are met and the well-posedness is proved.

We wish to approximate this problem with the $\mathbb{P}_1$ finite element on a uniform mesh with step $h = 1/(N+1)$. We denote by $(\phi_1, \ldots, \phi_N)$ the classical hat function basis of $V_h = X_h^1 \cap H_0^1(\Omega)$. The approximate solution u_h is solution to the problem: find $u_h \in V_h$ such that for all $v_h \in V_h$

$$a(u_h, v_h) = \int_0^1 v_h \tag{4}$$

5) Introduce the Peclet number:

$$\gamma = \frac{h\beta}{\nu}$$

and compute the matrix associated to problem (4).

$$\left[\int_0^1 \phi_i' \phi_j'\right] = \frac{1}{h} \begin{bmatrix} 2 & -1 & & & (0) \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ (0) & & & -1 & 2 \end{bmatrix}$$

and

$$\left[\int_0^1 \phi_i' \phi_j\right] = \begin{bmatrix} 0 & \frac{1}{2} & & & (0) \\ -\frac{1}{2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{2} \\ (0) & & & -\frac{1}{2} & 0 \end{bmatrix}$$

The matrix associated to the problem is therefore:

$$A = \frac{\nu}{h} \begin{bmatrix} 2 & -1+\frac{\gamma}{2} & & & (0) \\ -1-\frac{\gamma}{2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1+\frac{\gamma}{2} \\ (0) & & & -1-\frac{\gamma}{2} & 2 \end{bmatrix}$$

6) Simulations in Matlab. Take $\beta = 1$, $h = 0.01$ and plot the solutions to (4) for values of ν corresponding to $\gamma = 1$ and $\gamma = 5$. Try different values of γ to determine an instability threshold. What is the meaning of this threshold for the matrix of the problem?

The right-hand side vector is

$$\left[\int_0^1 \phi_i\right] = \begin{bmatrix} h \\ \vdots \\ \vdots \\ h \end{bmatrix}$$

The solutions for values of ν corresponding to $\gamma = 1$ and $\gamma = 5$ are reported in Figure 2.

The instability threshold is found for $\gamma^\star = 2$.

REMARK: This corresponds to the value of γ for which the matrix is no longer an M-matrix. The discrete maximum principle is violated.

Definition: A matrix $A = [a_{i,j}] \in \mathbb{R}^{n\times n}$ is said to be an M-matrix if

- It is invertible and all the entries of A^{-1} are non-negative.
- All its non-diagonal elements $a_{i,j},\ i \neq j$ are non-positive.

Property 1: (Continuous maximum principle) The maximum principle for the advection-diffusion equation $-\nu u'' + \beta u = f$ on a bounded domain $x \in \Omega$ holds, that is:

$$f(x) \geq 0 \Rightarrow u(x) \geq 0.$$

Property 2: (Discrete maximum principle) Let $A_h U_h = b_h$ be a linear system associated with a $\mathbb{P}_1$ finite element formulation of the advection diffusion $-\nu u'' + \beta u = f$ with $f(x) \geq 0$. Then, if A_h is an M-matrix, the solution u_h of the associated variational formulation satisfies $u_h(x) \geq 0$.

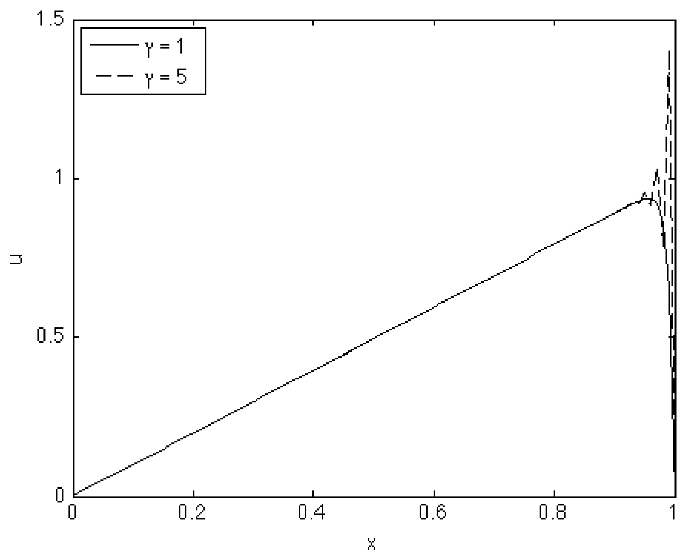

Figure 2: Numerical solution of the Galerkin-based $\mathbb{P}_1$ Finite Element formulation for various values of γ

Proof: Noticing first that $f \geq 0$, $b_h^j = \int_0^1 f\phi_j dx \geq 0$. Furthermore, A_h is invertible and since it is an M-matrix

$$U_h = A_h^{-1} b_h \geq 0.$$

$u_h \geq 0$ holds since $u_h(x) = \sum_{j=1}^{n} U_h^j \phi_j(x)$.

A similar inequality exists for the upper bound of the right-hand side, ensuring that the solution remains bounded, and large oscillations cannot occur.

7) Let $b : [0,1] \to \mathbb{R}$ be a smooth function such that $b(0) = b(1) = 0$ and $b(\xi) > 0$ on $(0,1)$ (b is called a *bubble function*). We propose to replace the test function space V_h by another space W_h, while keeping the same space V_h to look for the solution. We define W_h spanned by the functions ψ defined by:

$$\psi_i = \phi_i + \begin{cases} b\left(\dfrac{x - x_{i-1}}{h}\right), & x \in [x_{i-1}, x_i], \\ -b\left(\dfrac{x - x_i}{h}\right), & x \in [x_i, x_{i+1}], \end{cases}$$

where $x_i = ih$ is the i^{th} node of the mesh. Choose an arbitrary bubble function b and plot the basis function ψ_i. Comment.

Two arbitrary bubble functions are chosen here:

$$b_1(x) = x(1-x), \quad b_2(x) = \sin(\pi x)$$

The two corresponding basis functions are reported in Figure 3 as well as ϕ_i. The functions ψ_i give more weight "upstream" than "downstream". (here $\beta > 0$).

8) Consider the problem: search for $\bar{u}_h \in V_h$ such that

$$a(\bar{u}_h, v_h) = \int_0^1 v_h, \qquad \forall v_h \in W_h \tag{5}$$

Compute the matrix associated to this problem. Remark: The approach consisting of choosing different spaces for the solution and the test functions is referred to as the *Petrov-Galerkin method.*

$$\bar{A} = \left[\nu \int_0^1 \phi_j' \psi_i' + \beta \int_0^1 \phi_j' \psi_i\right]$$

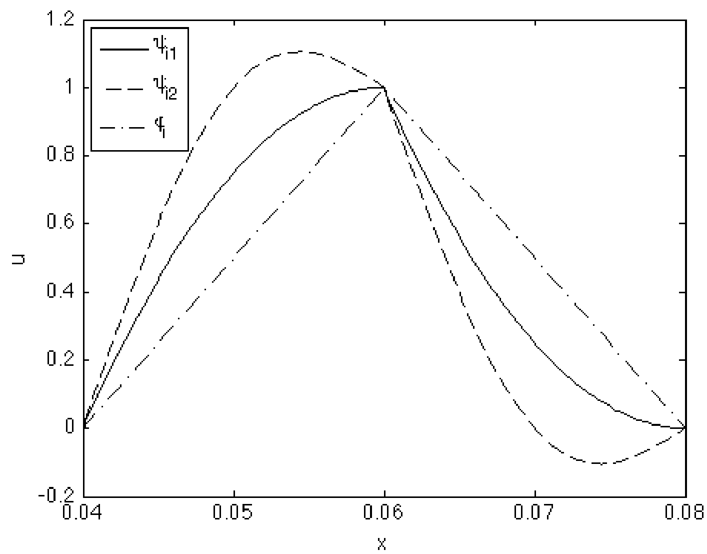

Figure 3: Respective basis functions corresponding to the two bubble functions b_1 and b_2,

- The diffusive terms are not affected by the bubble function:

$$\int_0^1 \phi_j' \psi_i' = \pm\frac{1}{h}\int_0^1 (\phi_i' \pm b') = \int_0^1 \phi_j'\phi_i' + \frac{1}{h}\int_0^1 b' = \int_0^1 \phi_j'\phi_i' + 0$$

- The advective terms are modified.

The matrix associated to the problem is then:

$$\bar{A} = \begin{bmatrix} 2\frac{\nu_h}{h} & \frac{\beta}{2} - \frac{\nu_h}{h} & & & (0) \\ -\frac{\beta}{2} - \frac{\nu_h}{h} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{\beta}{2} - \frac{\nu_h}{h} \\ (0) & & & -\frac{\beta}{2} - \frac{\nu_h}{h} & 2\frac{\nu_h}{h} \end{bmatrix}$$

with

$$\nu_h = \nu + \beta h \int_0^1 b(x)dx$$

Defining a Peclet number $\gamma_h = \frac{h\beta}{\nu_h}$, the matrix can be written as

$$\bar{A} = \frac{\nu_h}{h}\begin{bmatrix} 2 & -1 + \frac{\gamma_h}{2} & & & (0) \\ -1 - \frac{\gamma_h}{2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 + \frac{\gamma_h}{2} \\ (0) & & & -1 - \frac{\gamma_h}{2} & 2 \end{bmatrix}$$

9) Show that the matrix resulting from the Petrov-Galerkin method (5) **is the same as that resulting from the Galerkin method** (4) **up to the modification of the diffusion coefficient. Show that problem** (5) **is equivalent to searching for** $\bar{u}_h \in V_h$ **such that**

$$a_h(\bar{u}_h, v_h) = \int_0^1 v_h, \qquad \forall v_h \in V_h \tag{6}$$

where a_h **is a bilinear form which depends on** h**.**

$$a_h(u_h, v_h) = \nu_h \int_0^1 u_h' v_h' + \beta \int_0^1 u_h' v_h$$

10) How to choose $\int_0^1 b$ as a function of γ such that problem (6) is always stable.

In order for the matrix to be an M-matrix, a necessary condition is:

$$\frac{\beta}{2} - \frac{\nu_h}{h} < 0$$

that is

$$\int_0^1 b > \frac{1}{2} - \frac{1}{\gamma}$$

11) Simulation in Matlab. Take $\beta = 1$, $h = 0.01$, a bubble function of your choice. Plot the solution to (6) for $\gamma = 1$ and $\gamma = 5$. Comment.

The right-hand side vector is still:

$$\left[\int_0^1 \psi_i\right] = \begin{bmatrix} h \\ \vdots \\ \vdots \\ h \end{bmatrix}$$

The bubble function b_2 verifies

$$\int_0^1 b_2 = \frac{2}{\pi} > \frac{1}{2} - \frac{1}{5}$$

The corresponding solutions are reported in Figure 4. There are no more oscillations for $\gamma = 5$.

12) Prove and comment the following error estimate:

$$|u - \bar{u}_h|_1 \leq \left(1 + \frac{\|a\|}{\nu}\right) \inf_{w_h \in V_h} |u - w_h|_1 + \frac{1}{\nu} \inf_{w_h \in V_h} \sup_{v_h \in V_h} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{|v_h|_1}$$

Hint: First notice that, generally speaking, if $a(\cdot,\cdot)$ is α-coercive, one always have:

$$\alpha \|u\|_X \leq \sup_{v \in X} \frac{a(u, v)}{\|v\|_X}$$

Next prove:

$$\nu |u_h - w_h|_1 \leq \sup_{v_h \in V_h} \frac{a_h(u_h - w_h, v_h)}{|v_h|_1}$$

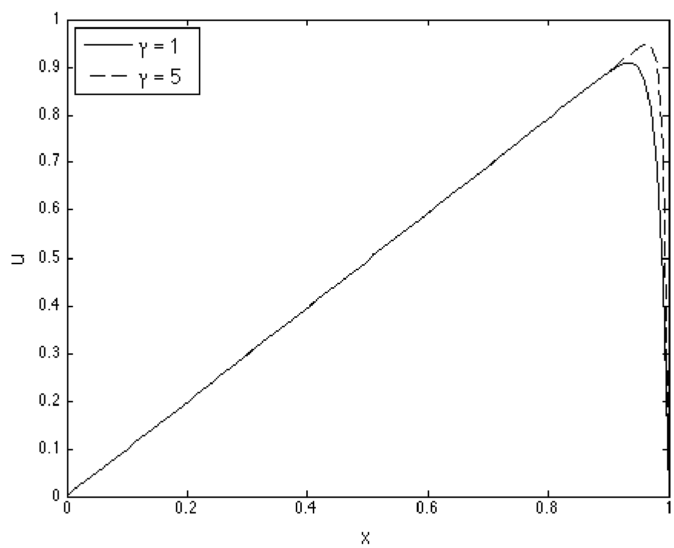

Figure 4: Numerical solution of the Petrov-Galerkin-based $\mathbb{P}_1$ Finite Element formulation for various values of γ

and

$$a_h(\bar{u}_h - w_h, v_h) = a(u - w_h, v_h) + a(w_h, v_h) - a_h(w_h, v_h)$$

If a is α-coercive

$$\alpha \|u\|_X^2 \leq a(u, u)$$

implies that

$$\alpha \|u\|_X \leq \frac{a(u, u)}{\|u\|_X} \leq \sup_{v \in X} \frac{a(u, v)}{\|v\|_X}$$

Since:

$$\begin{aligned} a_h(v_h, v_h) &= \nu_h \int_0^1 v_h'^2 \\ &\geq \nu |v_h|_1^2 \end{aligned}$$

a_h is ν-coercive for the norm $|\cdot|_1$. Therefore, if $u_h, w_h \in V_h$:

$$\nu |u_h - w_h|_1 \leq \sup_{v_h \in V_h} \frac{a_h(u_h - w_h, v_h)}{|v_h|_1}$$

Moreover

$$\begin{aligned} a_h(\bar{u}_h - w_h, v_h) &= a_h(\bar{u}_h, v_h) - a_h(w_h, v_h) \\ &= \langle 1, v_h \rangle - a_h(w_h, v_h) \\ &= a(u, v_h) - a_h(w_h, v_h) \\ &= a(u - w_h, v_h) + a(w_h, v_h) - a_h(w_h, v_h) \end{aligned}$$

Then for any $w_h \in V_h$

$$\begin{aligned} |u - \bar{u}_h|_1 &\leq |u - w_h|_1 + |w_h - \bar{u}_h|_1 \\ &\leq |u - w_h|_1 + \frac{1}{\nu} \sup_{v_h \in V_h} \frac{|a_h(\bar{u}_h - w_h, v_h)|}{|v_h|_1} \\ &\leq |u - w_h|_1 + \frac{1}{\nu} \sup_{v_h \in V_h} \frac{|a(u - w_h, v_h)| + |a(w_h, v_h) - a_h(w_h, v_h)|}{|v_h|_1} \\ &\leq |u - w_h|_1 + \frac{\|a\|}{\nu} |u - w_h|_1 + \frac{1}{\nu} \sup_{v_h \in V_h} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{|v_h|_1} \\ &\leq \left(1 + \frac{\|a\|}{\nu}\right) |u - w_h|_1 + \frac{1}{\nu} \sup_{v_h \in V_h} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{|v_h|_1} \end{aligned}$$

Since this is valid for any $w_h \in V_h$,

$$|u - \bar{u}_h|_1 \leq \inf_{w_h \in V_h} \left(\left(1 + \frac{\|a\|}{\nu}\right) |u - w_h|_1 + \frac{1}{\nu} \sup_{v_h \in V_h} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{|v_h|_1} \right) \tag{7}$$

The error is bounded by two terms: the first term is proportional to the distance from the the solution u to the subspace V_h and the second terms is the error due to Petrov-Galerkin formulation: the solution $\bar{u}_h$ verifies a variational formulation in a_h instead of a.