Assignment 4

(Due in class March 5, credit: 25%)

Consider the differential equation

$$u_t = u_x \ , \quad 0 \le x < \infty \, , \ 0 \le t \, ,$$

and the approximation

$$\begin{aligned} u_j^{n+1} - u_j^n &= \frac{k}{2} D_0 (u_j^{n+1} + u_j^n) \, , \quad j = 1, 2, \dots \, , \\ u_0^{n+1} - u_0^n &= k D_+ u_j^n \, , \\ u_j^0 &= f_j \, , \\ ||u^n||_h &< \infty \, . \end{aligned} \tag{1}$$

1. Find the number λ_0 such that (1) is strongly stable if and only if $\lambda \le \lambda_0$, where $\lambda = k/h = const.$
2. Prove that the error $w_j^n = u_j^n - u(x_j, t_n)$ satisfies $||w^n||_h \le K h^2$.
3. Consider the problem in the finite interval $0 \le x \le 1$, and prescribe a boundary condition $u(1,t) = 1$. Write a program that solves the problem using (1) with the condition $||u^n|| < \infty$ replaced by

   $$u_N^n = 1 \, ,$$

   and with the initial function

   $$f(x) = 1 + 100 x^4 (x-1)^4 \, .$$

   Run the scheme for $\lambda = \lambda_0 - 0.1$ and $\lambda = \lambda_0 + 0.1$, and demonstrate the stability properties by plotting u_0^n for $0 \le t_n \le 1 \, , \ N = 100$.
4. Find the true solution $u(x, 0.5)$ and compute the error $w_j^n = u_j^n - u(x_j, t_n)$ for $\lambda = 1$ and $t_n = 0.5/k$. Measure the norm

   $$||w^n||_h = \Big(\sum_{j=0}^{N-1} |w_j^n|^2 h \Big)^{1/2}$$

   for the two cases $N = 100$ and $N = 200$, and show that the error is of second order in h.