Chapter 1

What this course is about

This course is a Ph.D. level introduction to the key ideas and concepts related to the mathematical study of uncertainty. This area is known as “probability” or “stochastics.”

There is no area of engineering, the physical sciences, economics, finance or the social sciences that has not been profoundly impacted by the use of stochastic methods. In addition, knowledge of this subject matter will change the way one approaches the formulation of mathematical and computational models. Indeed, knowledge of probability offers a new “language” for addressing model formulation.

Stochastics also has close connections with many other areas of computational and applied mathematics:

- linear algebra
- differential equations
- discrete mathematics

Even for areas with which one may already be familiar, knowledge of probabilistic ideas will add richness to one’s understanding. A good example is the theory of “least squares.” We will see that the statistical perspective on least squares allows one to develop a rigorous conceptual framework that can enhance one’s application of least squares (e.g. in choosing good “weights” for the application of weighted least squares).

Our discussion of the subject matter will focus heavily on how to convert stochastic problems into computations that can be addressed through numerical linear algebra or differential equations. We will also see that “stochastic simulation” (also know as Monte Carlo simulation) is a powerful means of doing computation in the stochastic modeling context. In addition, model building in the stochastic context often requires that we build models that respect observational data that has been collected. This model calibration issue requires that we also familiarize ourselves with the relevant statistical principles.

We end this discussion with a brief illustration of the rich connections between stochastics and other areas of applied mathematics. Consider a particle that randomly moves between the sites of $\{0, 1, \ldots, N\}$. See Figure 1.1 This “random walker”, if currently in site i, moves to its neighbors $i-1$ and $i+1$ with equal probability, $\frac{1}{2}$. The sites 0 and N are “absorbing sites,” so that once the particle hits either of those sites, it is “absorbed” into that site and never again leaves. (This is one version of the “gambler’s ruin” model, where the random walk corresponds to the time evolution of the gambler’s wealth.) One question that is of interest is the computation of $u(i) =$ the probability that the random walker is absorbed into site 0 (i.e. “ruined”) given that they start at site i. A little thought shows that by considering the likely location of the walker after one step, the $u(j)$’s should satisfy:

$$u(i) = \frac{1}{2}u(i-1) + \frac{1}{2}u(i+1) \tag{1.1}$$

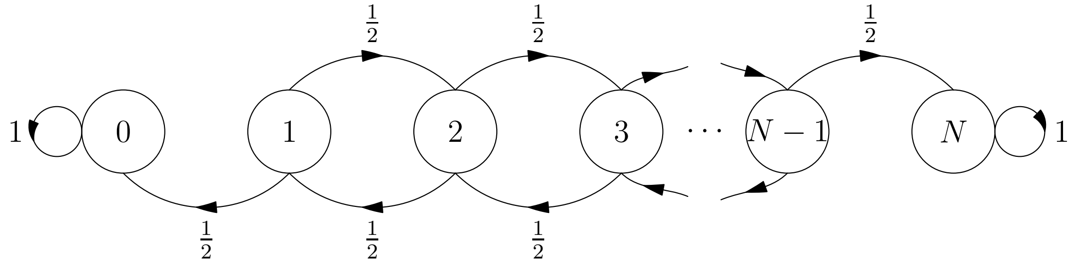

Figure 1.1: Markov Chain Corresponding to States with Transition Probabilities

subject to the "boundary conditions" $u(0) = 1$ and $u(N) = 0$.

Recall that for a general sequence $\{v(j) : j \in \mathbb{Z}\}$, we can define the "first difference" sequence as

$$(\Delta v)(j) = v(j) - v(j-1).$$

In general, the "k^{th} difference" sequence is then given by

$$(\Delta^k v)(j) = (\Delta^{k-1} v)(j) - (\Delta^{k-1} v)(j-1).$$

With this differencing notation, (1.1) can be rewritten as:

$$(\Delta^2 u)(i+1) = 0 \tag{1.2}$$

such that $u(0) = 1$ and $u(N) = 0$. This suggests that there should be a connection to the differential equation

$$\frac{d^2}{dx^2} u(x) = 0 \tag{1.3}$$

such that $u(0) = 1$ and $u(1) = 0$. This passage from the difference equation (1.2) to the differential equation (1.3) has to do, from a stochastic viewpoint, with the connection between a random walk and Brownian motion.