Due Date: This assignment is due on Thursday, 21 May, 2009(with the natural 1 day extension for SCPD students), by 5pm in the box outside Durand 112. The LaTeX incentive policy is in effect.

Problem 1: The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3, someone takes all the papers in the pile and puts them in the recycling bin. Also, if ever there are at least five papers in the pile, MR. Smith (with probability 1) takes all the papers to the bin.

This problem can be modeled with a 5 state Markov Chain, where the states are $\{0, 1, 2, 3, 4\}$ corresponding to the number of papers in the pile in the evening. The transition matrix is given below:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1. This problem can be modeled with a 5 state Markov Chain, where the states are $\{0, 1, 2, 3, 4\}$ corresponding to the number of papers in the pile in the evening. What is the corresponding transition matrix?

   Solution: Solving

   $$\pi P = \pi$$

   will give us the stationary distributions since the Markov Chain is aperiodic and irreducible.

   $$\pi = \begin{pmatrix} 0.3839 \\ 0.2559 \\ 0.1706 \\ 0.1137 \\ 0.0758 \end{pmatrix}$$

   The expected number of papers in the pile is:

   $$E\left[\# \text{ of papers in the pile}\right] = \sum_{j=0}^{4} j\pi_j = 1.2417$$

2. After a long time, what would be the expected number of papers in the pile?

   Solution:
   Solving

   $$\pi P = \pi$$

   will give u the stationary distributions since the Markov Chain is aperiodic and irreducible.

   $$\pi = \begin{pmatrix} 0.3839 \\ 0.2559 \\ 0.1706 \\ 0.1137 \\ 0.0758 \end{pmatrix}$$

   The expected number of papers in the pile is:

   $$E\left[\# \text{ of papers in the pile}\right] = \sum_{j=0}^{4} j\pi_j = 1.2417$$

3. Assume the pile starts with 0 papers. What is the expected time until the pile will again have 0 papers? **Solution:** We are asked to find the expected return time to state 0. This is given by:

$$E\left[\text{return time to state } 0\right] = \pi_0^{-1} = 2.6049$$

Problem 2: Let $G = (V, E)$ be a finite undirected and connected graph. Let $X = (X_n : n \geq 0)$ be a Markov chain that moves from vertex to vertex by choosing uniformly among the available edges. Compute the stationary distribution of this Markov chain (known as "random walk on the graph G")

Solution: The transition matrix has entries

$$P_{ij} = \frac{1}{\deg(v_i)} \mathbb{1}_{\{\{v_i, v_j\} \in E\}}$$

which is to say, the transition probability is uniform between neighbors.

From the fact that the graph is connected we know that there is a path from any vertex to any other vertex. Thus there is a non zero probability from going from any state (in the corresponding Markov chain) to any other (with probability given by the probability of that path). Thus the state is irreducible.

Note we do not assume that the resulting Markov chain is aperiodic, and thus interpret the steady state distribution as the long run expected percentage of time in a particular state.

We seek solutions to

$$\pi = \pi P.$$

Intuitively it makes sense that the long run time at a node should be proportional to how many neighbors it has. Namely we hypothesize that

$$\pi_i = \alpha \deg(v_i).$$

We use the face that

$$\sum_{i=1}^{|V|} \deg(v_i) = 2|E|$$

to guess that $\alpha = (2|E|)^{-1}$ so that

$$\pi_j = \frac{\deg(v_j)}{2|E|}.$$

We now verify that this solves $\pi = \pi P$

$$\frac{\deg(v_j)}{2|E|} = \pi_j = \sum_{i=1}^{|E|} \pi_i \, \mathrm{P}_{ij} = \frac{1}{2|E|} \sum_{i=1}^{|E|} \frac{\deg(v_i)}{\deg(v_i)} \mathbb{1}_{\{\{v_i, v_i\} \in E\}} = \frac{1}{2|E|} \sum_{i=1}^{|E|} \mathbb{1}_{\{\{v_i, v_j\} \in E\}} = \frac{\deg(v_j)}{2|E|},$$

since the last term counts the neighbors of v_j.

Problem 3: (Lawler, Problem 1.8) Consider a simple random walk on the graph shown in Figure 1. (Recall that a simple random walks on a graph is the Markov chain which at each time moves to an adjacent vertex, each adjacent vertex having the same probability.)

The transition probabilities are given by

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

1. In the long run, about how much time is spent in vertex A?

   Solution:
   We solve the stationary distribution system

   $$\pi P = \pi$$

   such that

   $$\sum_j \pi_j = 1 \qquad \text{and} \qquad 0 \leq \pi_j \leq 1 \quad \forall j$$

   We find that:

   $$\pi = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

   A walker will spend on average $\pi_A = 25\%$ of his/her time in state A.

2. Suppose a walker starts in vertex A. What is the expected number of steps until the walker returns to A?

   Solution:
   The expected return to a state j is given by π_j^{-1} Therefore, the expected return time to A is 4.

3. Suppose a walker starts in vertex C. What is the expected number of visits to B before the walker reaches A?

   Solution:
   Let U_j be the expected number of visits to B before A, starting at state i. Thus:

   $$U_A = 0 \quad U_B = P_{BA} \times 0 + P_{BC}U_C + P_{BE}U_E \quad U_C = P_{CA} \times 0 + P_{CB}(1 + U_B)$$

   $$U_D = P_{DA} \times 0 + P_{DE}U_E \quad U_E = P_{ED}U_D + P_{EB}(1 + U_B)$$

   This leads to the following linear system for $\vec{U}$:

   $$\begin{pmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} U_B \\ U_C \\ U_D \\ U_E \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

   Solving this we find that:

   $$\vec{U} = \begin{pmatrix} 0.6364 \\ 0.8182 \\ 0.5455 \\ 1.0909 \end{pmatrix}$$

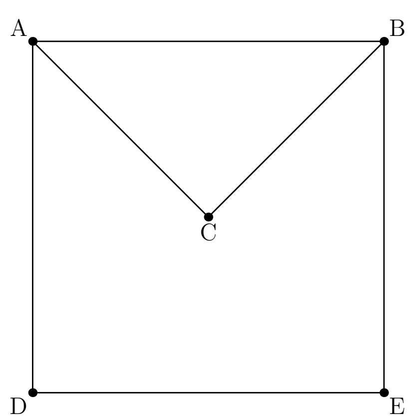

Figure 1: Graph for Problem 3

Therefore the expected number of visits starting in state C, to state B before one arrives at state A is: $U_C = 0.8182$.

4. Suppose a walkers starts in vertex B. What is the probability that the walker reaches A before the walker reaches C?

Solution:
Let P_j be the probability that a walker reaches A before C given that they are in state j. Therefore:

$$P_A = 1 \qquad P_B = P_{BA} \times 1 + P_{BC} \times 0 + P_{BE} \times P_E \qquad P_C = 0$$

$$P_D = P_{DA} \times 1 + P_{DE} P_E \qquad P_E = P_{ED} = P_D + P_{EB} P_B$$

This leads to the system:

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} P_B \\ P_D \\ P_E \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

Solving this and normalizing the solution we find:

$$\vec{P} = \begin{pmatrix} 0.5714 \\ 0.8571 \\ 0.7143 \end{pmatrix}$$

Therefore the probability of reaching A before C given that you start in state B is $P_B = 0.5714$.

5. Again, assume the walker starts in C. What is the expected number of steps until the walker reaches A?

Solution:
Let U_j be the expected number of steps from state j to state A. Thus:

$$U_A = 0 \qquad U_B = 1 + P_{BA} + P_{BC} U_C + P_{BE} U_E \qquad V_C = 1 + P_{CA} + P_{CB} U_B$$

$$U_D = 1 + P_{DA} + P_{DE} U_E \qquad U_E = 1 + P_{ED} P_D + P_{EB} + U_B$$

This gives the following linear system:

$$\begin{pmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} U_B \\ U_C \\ U_D \\ U_E \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Solving this gives:

$$\vec{U} = \begin{pmatrix} 3.2727 \\ 2.6364 \\ 3.0909 \\ 4.1818 \end{pmatrix}$$

Therefore the expected number of steps from C to A is $U_C = 2.6364$.

Problem 4: (Bremaud, Problem 3.3.5) A knight moves randomly on a chessboard, making each admissible move with equal probability, and starting from a corner. What is the average time he takes to return to the corner he started from?

Solution: by Chang-han Rhee
Chessboard has 64 cells. Each cell can be seen as a vertex of a finite graph with each edge connecting a pair of cells the night can move between. We count the degree of vertices(cells) as follows:

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

We can see that the total degree is 336 and each corner has degree 2. Applying the result of Problem 2, we can see that the expected time the knight comes back to the corner is $\frac{336}{2} = 168$.

Problem 5: Choice Theory: Markov Chain Models of Brand Switching

Suppose for a given product, a consumer has a choice between m competing products, labeled $1, 2, \ldots, m$. Suppose successive purchases by a consumer are modeled as an ergodic Markov chain $\{X_n\}$ with transition matrix $P = \{p_{ij}, 1 \le i, j, \le m\}$.

The diagonal entries of P are measures of consumer loyalty to a brand since p_{jj} measures the tendency of a consumer to repurchase brand j when his last purchase was brand j.

Companies are interested in market share. The market share of a brand j is the percentage of purchases made which are of brand j. Assuming a homogeneous population, the market share of brand j can be evaluated as

$$\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N} \mathbb{1}_{\{X_n = j\}},$$

which we know is just $\pi(j)$, where π is the stationary distribution for the chain.

1. Consider a simple case. Suppose $\forall j$, $p_{jj} = 1/2$ and $\forall i \neq j$, $p_{ij} = (2(m-1))^{-1}$. Check that this matrix is doubly stochastic and find its stationary distribution.
2. Now assume that for $i \neq j$,

$$p_{ij} = \frac{1 - p_{ii}}{m-1},$$

so that a consumer not choosing his old brand chooses uniformly among the others. With this simple specification of choice probabilityies, can we see the dependence of market share on consumer loyalty? Verify

$$\pi(j) = \frac{(1 - p_{jj})^{-1}}{\sum_{k=1}^{m} (1 - p_{kk})^{-1}},$$

so that $\pi(j) \uparrow 1$ as $p_{jj} \uparrow 1$. (Wolf, 1989)

Solution: by Christopher Maes

1. A transition matrix P is doubly stochastic if

$$\textstyle\sum_{j=1}^{m} p_{ij} = 1 \quad \forall i$$

From the definition of p_{ij} we have

$$\sum_{j=1}^{m} p_{ij} = \frac{1}{2} + \sum_{j\neq i} \pi_i \frac{1}{2\,(m-1)} = \frac{1}{2} + \frac{m-1}{2\,(m-1)} = 1$$

Thus the matrix is doubly stochastic. If a transition matrix is doubly stochastic each p_{ii} is in the form $\pi_i = 1/N$ where N is the number of states. Thus the stationary distribution is

$$\pi_i = \tfrac{1}{m} \quad \forall i$$

2. The stationary distribution is the solution to $\pi = \pi P$ and $\sum \pi = 1$. Expanding the first relation we have

$$\begin{array}{rcl} \pi_j & = & \sum_1^m \pi_i P_{ij} = \sum_{i \neq j} \pi_i \frac{1-p_{ii}}{m-1} + \pi_j p_{jj} \\ \pi_j \left(1 - p_{jj}\right) & = & \sum_{i \neq j} \pi_i \frac{1-p_{ii}}{m-1} \\ \pi_j & = & \frac{\left(1-p_{jj}\right)^{-1}}{\sum_{i \neq j} \pi_i \frac{1-p_{ii}}{m-1}} \end{array}$$

From here, two strategies are possible:

- We can apply the second equation $\sum \pi = 1$ and solve for π.
- Or we can verify that the given π satisfy those equations.

Thus we have an expression for the denominator $\sum_{i \neq j}$ and the distribution π_j is given by

$$\pi_j = \frac{\left(1 - p_{jj}\right)^{-1}}{\sum_{k=1}^m \pi_i \left(1 - p_{kk}\right)^{-1}}$$

From this expression we can see that as $p_{jj} \to 1$ that the denominator of the above expression looks like $\left(1 - p_{jj}\right) / \left(1 - p_{jj}\right) \to 1$ and thus $j \to 1$.

Problem 6: Non-Parametric Maximum Likelihood Estimation of Transition Probabilities

Imagine we have a Markov chain $\{X_n\}$ on the finite state space $\mathbb{S}$ with the number of states m known and the transition $P = (p_{i,j}, 1 \leq i, j \leq m)$ matrix unknown. We observe the chain during the times $0, 1, \ldots, n$ and wish to estimate P.

If $x_0, \ldots, x_n$ is the realization of the Markov chain to time n, then the likelihood function $L_n(P)$ is given by

$$\begin{aligned} L_n(P) &= \mathbf{P}(X_0 = x_0, \ldots, X_n = x_n) \\ &= \mathbf{P}(X_0 = x_0) p_{x_0,x_1} \cdots p_{x_{n-1},x_n}. \end{aligned}$$

Setting $a_{x_0} = \mathbf{P}(X_0 = x_0)$, the log-likelihood function $\mathcal{L}_n(P) = \log L_n(P)$ is then given by

$$\begin{aligned} \mathcal{L}_n(P) &= \log(a_{x_0} p_{x_0,x_1} \cdots p_{x_{n-1},x_n}) \\ &= \log \left(a_{x_0} \prod_{(i,j) \in \mathbb{S} \times \mathbb{S}} p_{i,j}^{N_{i,j}} \right) \\ &= \log a_{x_0} + \sum_{(i,j) \in \mathbb{S} \times \mathbb{S}} N_{i,j} \log p_{i,j}, \end{aligned}$$

where

$$N_{i,j} = \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k = i, X_{k+1} = j\}}.$$

Define

$$N_i = \sum_{k \in \mathbb{S}} N_{ik}.$$

Show that the maximum likelihood estimator of P is given by

$$\hat{p}_{i,j} = \frac{N_{i,j}}{N_i}.$$

Solution: by Nick West

We are now solving a constrained optimization problem:

$$\begin{array}{rcl} \mathcal{L}_n(P) & = & \log a_{x_0} + \sum_{(i,j) \in \mathbb{S} \times \mathbb{S}} N_{i,j} \log p_{i,j} \\ g_k\left(P\right) & := & \sum_{j=1}^m p_{k,j} = 1, \ k = 1, \ldots, m \end{array}$$

We introduce one Lagrange multiplier λ_i for each constraint require that $\sum_{j=1}^m p_{k,j} = 1$ for all k. Equating derivatives we find that:

$$\frac{\partial \mathcal{L}_n(P)}{\partial p_{i,j}} = \frac{N_{i,j}}{p_{i,j}} = \lambda_i = \sum_k \left(\lambda_k \frac{\partial g_k(P)}{p_{i,j}} \right)$$

or

$$p_{i,j} = \frac{N_{i,j}}{\lambda_i}$$

Applying the constraints that the rows of P must sum to one, we find:

$$1 = \sum_j p_{i,j} = \sum_j \frac{N_{i,j}}{\lambda_i} = \frac{N_i}{\lambda_i} \Rightarrow \lambda_i = N_i$$

We find:

$$p_{i,j} = \frac{N_{i,j}}{N_i}.$$

Problem 7: The page-rank algorithm computes the equilibrium distribution of a finite state Markov chain having a transition matrix of the form

$$P = \alpha\Lambda + (1-\alpha)Q, \quad 0 < \alpha < 1,$$

where Λ is a rank-one stochastic matrix having identical positive rows and Q is an arbitrary stochastic matrix.

1. Compute the equilibrium distribution $\pi(\alpha)$ of P (in terms of something involving the inverse of a matrix).
2. How would you compute the sensitivity $\pi'(\alpha)$?

Solution:

1. Assume Λ has the identical rows λ. Observe that if $\pi(\alpha)$ exists, then $\pi(\alpha)\Lambda = \lambda$. Be the definition,

$$\pi(\alpha) = \pi(\alpha) P = \pi(\alpha)(\alpha\Lambda + (1-\alpha) Q) = \alpha\lambda + (1-\alpha)\pi(\alpha) Q$$

   We have

$$\pi(\alpha)(I - (1-\alpha) Q) = \alpha\lambda$$

   Note that by the Gershgorin thm, the spectral radius of Q is less than or equal to 1. Therefore all eigenvalues of $I - (1-\alpha) Q$ are nonzero. It also means that $I - (1-\alpha) Q$ is nonsingular. Then

$$\pi(\alpha) = \alpha\lambda (I - (1-\alpha) Q)^{-1}.$$

2. Take the derivative $\pi(\alpha)(I - (1-\alpha) Q) = \alpha\lambda$ on both sides with respect to α. We have

$$\pi'(\alpha)(I - (1-\alpha) Q) + \pi(\alpha) Q = \lambda$$

   so

$$\pi'(\alpha) = (\lambda - \pi(\alpha) Q)(I - (1-\alpha) Q)^{-1} = \lambda \left(I - \alpha (I - (1-\alpha) Q)^{-1} Q \right) (I - (1-\alpha) Q)^{-1}.$$