Due Date: This assignment is due on Thursday, 9 April, 2009, by 5pm in the box outside Durand 112. See the course website for the policy on incentives for LaTeX solutions.

Problem 1 (10 pts): You have two opponents with whom you alternate play. Whenever you play A, you win with probability p_A; when ever you play B, you win with probability p_B, where $p_B > p_A$. If your objective is to minimize the number of games you need to play in order to win two in a row, should you start with A or B?

Hint: Let $\mathbf{E}N_i$ denote the mean number of games needed if you initially play i. Derive an expression for $\mathbf{E}N_A$ that involves $\mathbf{E}N_B$; write down the equivalent expression for $\mathbf{E}N_B$ and then subtract.

Problem 2 (10 pts): A set of n dice is thrown. All those that land on six are put aside, and the others are again thrown. This is repeated until all the dice have landed on a six. Let N_n denote the number of throws needed. (For instance, suppose that $n = 3$ and that on the initial throw exactly two of the dice land on six. Then the other die will be thrown, and if it lands on a six, the $N_3 = 2$.) Let $m_n = \mathbf{E}N_n$.

1. Derive a recursive formula for m_n and use it to calculate m_i, $i = 2, 3, 4$ and to show that $m_5 \approx 13.024$.
2. Let X_i denote the number of dice rolled on the i^{th} throw. Find $\mathbf{E}\left[\sum_{i=1}^{N_n} X_i\right]$.

Problem 3 (10 pts): Consider the quadratic equation $x^2 + Bx + C = 0$ where B and C are independent and have uniform distributions on $[-n, n]$. Find the probability that the equations has real roots. What happens as $n \to \infty$?

Problem 4 (10 pts): Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed continuous random variables. We say that record at time n occurs if $X_n > \max\{X_1, \cdots, X_{n-1}\}$. That is, X_n is a record if it is larger than each of the previous X_i's.

Let

$$N = \min\{n : n > 1 \text{ and a record occurs at time } n\}.$$

Show $\mathbf{E}N = \infty$.

Problem 5 (10 pts): Compute the maximum likelihood estimators for a random sample of Beta (α^*, β^*) population.

Problem 6 (10 pts): Suppose we observe n independent samples of a random variable X, which has mean μ. We call X_i the i-th independent sample and denote the sample mean with $\hat{\mu}$.

Compute the mean of the following two quantities. Which is an unbiased estimator?

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$$

Problem 7 (10 pts): Suppose that f is a strictly positive continuous joint density of a random vector (X, Y) with $\mathbf{E}X^2 < \infty$.

1. Compute $\mathbf{E}\left(X | Y \in [y, y+h]\right)$
2. Compute $\phi(y) = \lim_{h\to 0}\mathbf{E}\left(X | Y \in [y, y+h]\right)$