This exam is open notes and open book. You have three hours. There are a total of 57 points. Good luck!

Problem 1 (16 pts): Suppose that we are testing a new product that will be marketed in six months. We start operating n prototypes of the product but will need to terminate the testing in four months (in order to deal with last-minute re-design issues, etc.). We model the test data as $T_1 \wedge t, T_2 \wedge t, \cdots, T_n \wedge t$, where T_i is the lifetime of prototype i (in months), $t = 4$ months, $a \wedge b = \min\{a, b\}$, and we assume, on the basis of past experience, that the T_i's are iid and exponentially distributed with (unknown) parameter λ^*. Suppose $n = 5$ and $T_i \wedge t = 1, 3, 4, 1, 2$ for $1 \le i \le 5$.

1. Compute the maximum likelihood estimator $\hat{\lambda}_5$. **(4 Points)**
2. Show that
$$\hat{\lambda}_n \overset{\mathcal{D}}{\approx} \lambda^* + n^{-\frac{1}{2}} \, \mathrm{N}\left(0, \sigma^2(\lambda^*)\right)$$
for large n and compute $\sigma^2(\lambda^*)$. **(6 Points)**
3. Use the result in part 2 to provide an approximate 95% confidence interval for λ^*. **(2 Points)**
4. Discuss how you would apply the bootstrap algorithm to compute a confidence interval for λ^*. **(4 Points)**

Problem 2 (10 pts): Suppose that the position of a particle is described by a one-dimensional standard Brownian motion process $B = (B(t) : t \ge 0)$ starting from $B(0) = 0$, with $\mathrm{E}\, B(t) = 0$ and $\mathrm{Var}\,(B(t)) = t$. Because of measurement discretization, we observe

$$Z_n = \lfloor B(n) \rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

1. Provide a formula for the conditional density of $B(n)$, conditional on $Z_0, \cdots, Z_n$. **(7 Points)**
2. Provide a recursive (in n) updating formula for the conditional density of the position $B(n)$, conditional on $Z_0, Z_1, \cdots, Z_n$. **(3 Points)**

Problem 3 (9 pts): In many communication networking settings, the integrated Ornstein-Uhlenbeck process is used as a model of the aggregate traffic to arrive by time t. In particular, suppose that

$$Z(t) = t + \int_0^t X(s)ds,$$

where $X = (X(t) : t \ge 0)$ is a stationary Gaussian process with $\mathrm{E}\, X(t) = 0$ and $\mathrm{Cov}\,(X(0), X(t)) = \sigma^2 e^{-t}$.

1. Compute the covariance function of Z. [Hint: Note that Z is affine in X.] **(3 Points)**
2. Does Z have differentiable trajectories? Why or why not? **(2 Points)**
3. Compute the conditional distribution of $Z(1/2)$, given that $Z(1) = z$. **(2 Points)**
4. Compute the conditional distribution of $Z(3/2)$, given that $Z(1) = z_1$ and $Z(2) = z_2$. **(2 Points)**

Problem 4 (12 pts): Suppose Ms. Smith owns a car but lives close enough to her job that she always walks whenever the weather is nice. Whenever it rains and the car is where Ms. Smith currently is, she will drive. On any given trip from / to work, it rains with probability p and is nice with probability $1 - p$, independent of the weather on the preceding trips.

1. We first wish to compute the long-run fraction of time that Ms. Smith gets wet.
   (a) Model this situation via a suitable discrete-time Markov chain. **(3 Points)**
   (b) Compute the long-run fraction of time that Ms. Smith gets wet. **(3 Points)**
2. Now, suppose that Ms. Smith can choose to drive if she has a car, even if it is nice out. Suppose that there are certain costs to driving or not driving under different circumstances (e.g. if you walk in the rain, you will have to take your clothes to the dry cleaner at a cost of ℓ; if you drive you must pay for the gas at a cost of $g < \ell$). For $\alpha > 0$, let

$$C = \sum_{i=0}^{\infty} e^{-\alpha i} c(X_i, a_i).$$

   (a) What equation would you solve to minimize

$$\mathrm{E}_x [C],$$

   (where x denotes starting at home (with the car, dry))? **(3 Points)**
   (b) Suppose we want to ensure that Ms. Smith gets wet only 20% of the time. How would you compute the optimal policy? **(3 Points)**

Problem 5 (10 pts):

1. Prove that when n is large

$$\pi \approx \frac{4}{n} \sum_{j=1}^{n} \mathbb{1}_{\{U_{j1}^2 + U_{j2}^2 \le 1\}}.$$

   where the U_{j1} and U_{j2} are all iid $\mathrm{U}(0,1)$ rvs. **(2 Points)**
2. What control variate would you use to decrease the variance of a Monte Carlo simulation of π? **(2 Points)**
3. Discuss how you would choose the control coefficient α. **(1 Point)**
4. Compute $\mathrm{E}[Y|U_1]$ where $Y = \mathbb{1}_{\{U_1^2 + U_2^2 \le 1\}}$ where U_1 and U_2 are independent $\mathrm{U}(0,1)$ rvs. **(2 Points)**
5. Show that for any rv Z such that $\mathrm{Var}(Z) < \infty$, and any other rv W, that

$$\mathrm{Var}(\mathrm{E}[Z|W]) \le \mathrm{Var}(Z).$$

   (2 Points)
6. Provide an alternative estimator for π, with lower variance than Y, based on parts 4 and 5 above. **(1 Point)**

Prof. Peter Glynn (glynn@stanford.edu)
Stanford University