CME308 Final Exam

3:30 PM

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This exam is open notes and open book. You have three hours. There are a total of 80 points. Good luck!

Problem 1 (15 points): Consider a single-server queue in which customers are served according to a last in / first out non-preemptive discipline. When a given customer completes service, the server begins processing the most recently arrived customer. Customers arrive at times 1,3,5,7,11 and 15. With corresponding service requirements 4,3,1,8 and 2.

1.1. How many customers are in the system at $t = 14.7$? **(7 points)**

Solution: This is the same problem as given in HW1. Please see that solution.

1.2. Suppose we run the model 6 times and there are 3, 4, 4, 5, 2 and 1 customers in the queue at time 14.7 for each run. Produce a 90% confidence interval for the expected number of customers in the system at time 14.7. **(4 points)**

Solution: One option is to use the Central Limit Theorem even though the sample size is small. The confidence interval will be the interval

$$\left[\bar{X} - \frac{zs}{\sqrt{n}}, \bar{X} + \frac{zs}{\sqrt{n}}\right]$$

where $n = 6$,

$$\bar{X} = n^{-1}\sum_{i=1}^{n} X_i \quad \text{and} \quad s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Another possibility is to use the standard boostrap procedure: Set $\alpha = \bar{X}$ and draw, with replacement the sets $\{X_{j1}, \cdots, X_{jn}\} = \{X\}_j$ for $j = 1, m$. Define

$$\alpha_j = n^{-1}\sum_{i=1}^{n} X_{ji}.$$

We now find z_1 and z_2 such that

$$m^{-1}\sum_{i=1}^{m} I(z_1 \leq \alpha_j - \alpha \leq z_2) \approx 1 - \delta.$$

The confidence interval is given by

$$[\alpha - z_2, \alpha + z_1].$$

1.3. Give an algorithm for generating random variables from the distribution function

$$F(z) = \begin{cases} 1 - \exp(-\lambda(z-1)^{\alpha}) & z \geq 1 \\ 0 & \text{o.w.} \end{cases}$$

(4 points)

Solution: We use the method of inversion to solve this problem. Note that

$$F^{-1}(y) = x = \left(\frac{\log(1-y)}{-\lambda}\right)^{\frac{1}{\alpha}} + 1.$$

Now we generate a uniform $[0,1]$ u and return $F^{-1}(u)$.

Problem 2 (10 points): Suppose that we are designing staffing levels for a marketing firm that uses home-based part-time employees to market their products over the phone. Each day, a random number, N, of employees will make time available to market the company's products, earning the company Y dollars in revenue. The company wishes to compute the average revenue per employee-day, denoted α. Assume that the sequence of random vectors $((Y_i, N_i) : i \geq 1)$ is i.i.d., where Y_i is the revenue on day i and N_i is the number of working employees on day i.

2.1. The quantity α is the limit, as $n \to \infty$, of

$$\frac{Y_1 + \cdots + Y_n}{N_1 + \cdots + N_n}.$$

Compute α in terms of the moments of Y and N. **(2 points)**

Solution: Recall that

$$\frac{Y_1 + \cdots + Y_n}{n} \to \mathrm{E}\left[Y\right] \quad \text{and} \quad \frac{N_1 + \cdots + N_n}{n} to \mathrm{E}\left[N\right].$$

Thus

$$\frac{Y_1 + \cdots + Y_n}{N_1 + \cdots + N_n} = \frac{Y_1 + \cdots + Y_n}{n} \frac{n}{N_1 + \cdots + N_n} \to \frac{\mathrm{E}\left[Y\right]}{\mathrm{E}\left[N\right]}.$$

2.2. If n is large, give a complete description of how you would compute a 95% confidence intervale for α based on $(Y_1, N_1), \cdots, (Y_n, N_n)$. **(8 points)**

Solution: We employ the bootstep method as follows: draw n samples with replacemnet from the set (Y_i, N_i). Define one bootstrap sample as

$$\alpha_j = \frac{Y_{j_1} + \cdots + Y_{j_n}}{N_{j_1} + \cdots + N_{j_n}}.$$

Now compute this quantity, say, m times. Now order the quantities $\hat{\alpha} - \alpha_j$ and compute the 5% and 95% quantiles, z_1 and z_2. The 90% confidence interval for α is

$$[\hat{\alpha} - z_1, \hat{\alpha} - z_2].$$

Problem 3 (10 points): In numerically simulating solutions $X = (X_t : t \geq 0)$ to stochastic differential equations, both first and second order schemes are available. If a k^{th} order scheme is used $(k = 1, 2)$, then an approximation X_h is simulated (involving simulating a discrete time approximation $X_h(0), X_h(h), X_h(2h), \cdots, X_h(nh), \cdots$) having a bias that satisfies

$$\mathrm{E}\left[f(X_h(1))\right] = \mathrm{E}\left[f(X(1))\right] + b_k h^k + \mathrm{o}(h^k)$$

as $h \downarrow 0$ for any f smooth (and for some constant b_k).

3.1. Suppose that we simulate n i.i.d. copies of the process X_h to time 1, thereby yielding $X_h^1(1), \cdots, X_h^n(1)$. We obtain an estimate

$$\frac{1}{n}\sum_{i=1}^{n} f(X_h^i(1))$$

for $\mathrm{E}\left[f(X(1))\right]$. What is the mean square error of this estimator for n large and h small? **(4 points)**

Solution: Define

$$Y_i = f(X_h^i(1)) \quad \text{and} \quad Y = f(X(1)).$$

We first compute the mean square error

$$\mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - \mathrm{E}\left[Y\right]\right)^2\right] = \mathrm{E}\left[\frac{1}{n^2}\left(\sum_{i=1}^{n} Y_i\right)^2 - 2\mathrm{E}\left[Y\right] n^{-1}\sum_{i=1}^{n} Y_i + \mathrm{E}\left[Y\right]^2\right] =$$

This simplifies to

$$= \frac{n\left(\mathrm{E}\left[Y_i^2\right] - \mathrm{E}\left[Y_i\right]^2\right) + n^2\left(\mathrm{E}\left[Y_i\right] - \mathrm{E}\left[Y\right]\right)^2}{n^2} = \frac{1}{n}\operatorname{var}(Y_i) + o(h^k) \approx \frac{\operatorname{var}(Y_i)}{n}.$$

Now we note that

$$\operatorname{var}(Y_i) = \operatorname{var}(Y) + b_k h^k(1 - 2\mathrm{E}\left[Y\right]) + o(h^k).$$

3.2. The total computation time required to simulate n such copies is roughly of order $c = n/h$. Subject to $n/h = c$,, what is the optimal allocation of n and h? What is the associated root mean square error convergence rate for a k^{th} order scheme under the optimal selection of n and h? **(6 points)**

Solution: We set $n = ch$ and find an equation with only h.

$$\text{MSE} = \frac{\operatorname{var}(Y) + b_k h^k(1 - 2\mathrm{E}\left[Y\right]) + o(h^k)}{ch} \approx \frac{\operatorname{var}(Y) + b_k h^k(1 - 2\mathrm{E}\left[Y\right])}{ch}$$

We now optimize over h and find the optimal h to be

$$h^* \approx \left(\frac{\operatorname{var}(Y)}{(k-1)b_k(1 - 2\mathrm{E}\left[Y\right])}\right)^{\frac{1}{k}}.$$

The optimal MSE becomes

$$\text{MSE} = \frac{\operatorname{var}(Y) + \left(\frac{\operatorname{var}(Y)}{(k-1)b_k(1-2\mathrm{E}[Y])}\right)^{\frac{k-1}{k}}(1 - 2\mathrm{E}\left[Y\right])}{c} = \frac{A + ((k-1)b_k)^{-\frac{k-1}{k}}B}{c}.$$

Problem 4 (14 points): Suppose that we run a newsstand that sells the *New York Times*. Assume that the demand on day i is Poisson distributed with unknown parameter λ^* and i.i.d. over time. (If your solution to a given part involves an optimization problem, just formulate it. Do not solve!)

4.1. Assume you observe the full demand over 5 days: 300, 200, 700, 400 and 800. What is your estimate $\hat{\lambda}$ for λ^*? **(4 points)**

Solution: We have $X_1 = 300$, $X_2 = 200$ etc. The likelihood of the these observations with some parameter λ is

$$L(\lambda) = \prod_i \frac{e^{-\lambda}\lambda^{X_i}}{X_i!}.$$

Thus the MLE for λ is

$$\lambda = \operatorname{argmax}_\lambda L(\lambda).$$

4.2. Suppose that you order 500 newspapers each day. You sell 400, 300, 500, 500 and 500 newspapers over 5 days and do not observe the unmet demand. How would you now estimate λ^*? **(5 points)**

Solution: Note that for the observations of 500 we need to replace the likelihood of a particular observation with the likelihood of X_i being greater than or equal to 500. Define

$$P_i(\lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^{X_i}}{X_i!} & X_i < 500 \\ \sum_{k=500}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} & X_i \geq 500 \end{cases}.$$

In this case we now need to maximize the likelihood function

$$\prod_i P_i(\lambda).$$

4.3. (continuation of 4.1) Suppose that you believe that demand is increasing linearly in time, so that the demand on day i is Poisson distributed with parameter $\lambda_i^* = a^* i + b^*$ for unknown a^* and b^*. The demand over 5 days is 300, 500, 400, 800 and 900 newspapers respectively. How would you estimate a^* and b^*? **(5 points)**

Solution: In this case the parameter λ depends on i so that

$$\lambda(i) = ai + b.$$

Plugging this into the result from the first question and have

$$L(a,b) = \prod_i \frac{e^{ai+b}(ai+b)^{X_i}}{X_i!}.$$

We now wish to maximize this over a and b.

Problem 5 (11 points): A spider hunting a fly moves between locations 1 and 2 according to a Markov chain with transition matrix P_s starting in location 1. The fly, unaware of the spider, starts in location 2 and moves according to a Markov chain with transition matrix P_f. The spider catches the fly and the hunt ends whenever they meet in the same locations.

$$P_s = \begin{bmatrix} 07 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} \qquad P_f = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$$

Show that the progress of the hunt, except for knowing the location where it ends, can be described by a three-state Markov chain where one absorbing state represents hunt ended and the other two that the spider and fly are at different locations.

5.1. Obtain the transition matrix for this chain. **(4 points)**
We note that the three states of interest are: Spider in state 1, Fly in state 2; Spider in state 2, Fly in state 1; both in the same state (end game). Now the transition matrix is give by

$$P = \begin{pmatrix} P_s(1,1)P_f(2,2) & P_s(1,2)P_f(2,1) & 1 - P_s(1,1)P_f(2,2) - P_s(1,2)P_f(2,1) \\ P_s(2,1)P_f(1,2) & P_s(2,2)P_f(1,1) & 1 - P_s(2,1)P_f(1,2) - P_s(2,2)P_g(1,1) \\ 0 & 0 & 1 \end{pmatrix}$$

where the states are order as above.

5.2. Find the probability that at time n the spider and fly are both at their initial locations. **(3 points)**

Solution: Let s be the initial state. Then the probability of being is the initial state at time n is

$$P^n(s,s).$$

5.3. What is the average duration of the hunt. **(4 points)**

We solve this be using first transition analysis. Let $u(x)$ be the expected length of the game given that we are in state x, for $x \neq E$. We see that for $x \neq E$ the following holds:

$$u(x) = 1 + \sum_{y\in S} P(x,y)u(y),$$

where $u(E) = 0$.

If you are short on time please specify the equations you would solve in each case before computing them.

Problem 6 (20 points): Let $X = \{X_t\}_{t\geq 0}$ be the solution to a stochastic differential equation of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t.$$

Let $\alpha > 0$ and $r : \mathbb{R} \to [0,\infty)$.

6.1. Suppose that we wish to compute

$$\mathrm{E}_x\left[\int_0^T \exp^{-\alpha t} r(X_t)dt\right]$$

where $T = \inf\{t \geq 0 : |X_t - x| \geq b\}$. What ODE would you solve? (Include any necessary boundary conditions.) **(4 points)**

Solution: For the following problem we interpret the stopping time as the following: give me an x and I'll define the $C = [x-b, x+b]$, which is the region the stopping time is associated with. Now

$$u(x) = \mathrm{E}_x\left[\int_0^T e^{-\alpha t} r(X_t)dt\right]$$

using iterated expectation we have

$$= \mathrm{E}_x\left[\mathrm{E}\left[\int_0^T e^{-\alpha t} r(X_t)dt | X(u) : 0 \leq u \leq h\right]\right]$$
$$= \mathrm{E}_x\left[\mathrm{E}\left[\int_0^h e^{-\alpha t} r(X_t)dt + \int_h^T e^{-\alpha t} r(X_t)dt | X(u) : 0 \leq u \leq h\right]\right]$$

as h is small we approximate the first term by its Taylor expansion giving

$$\approx hr(x) + \mathrm{E}_x\left[\mathrm{E}\left[\int_h^T e^{-\alpha t} r(X_t)dt | X(u) : 0 \leq u \leq h\right]\right]$$
$$= hr(x) + \mathrm{E}_x\left[e^{-\alpha h}\mathrm{E}\left[\int_0^{T-h} e^{-\alpha u} r(X_{u+h})du | X(u) : 0 \leq u \leq h\right]\right]$$

Now we use the Markov property

$$\approx hr(x) + \mathrm{E}_x\left[(1-\alpha h)\mathrm{E}\left[\int_0^{T-h} e^{-\alpha u} r(X_{u+h})|X(h)\right]\right]$$

We now note that the second expectation is the same path functional as we started with, just starting at the position $X(h)$ and running until the associted stopping time $\hat{T} = T - h$, which gives:

$$\begin{aligned}
&= hr(x) + \mathrm{E}_x\left[(1-\alpha h)\mathrm{E}_{X(h)}\left[\int_0^{\hat{T}} e^{-\alpha u} r(X_s) ds\right]\right] \\
&= hr(x) + (1-\alpha h)\mathrm{E}_x\left[u(X(h))\right]
\end{aligned}$$

Define the operator (generator)

$$Ł = \mu(x)\frac{\partial}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2}{\partial x^2}.$$

Now note that

$$\mathrm{E}_x\left[u(X(h))\right] = \mathrm{E}_x\left[u(x) + u_x(x)(X(h)-x) + \frac{u_{xx}(x)}{2}(X(h)-x)^2 + \cdots\right] = u(x) + h(Lu)(x).$$

Where the last equality follows from the lecture notes. Thus

$$u(x,t) = hr(x) + (1-\alpha h)(u + hŁu)$$

Now, as we subtract $u(x)$ from both sides, dividing h and sending $h \downarrow 0$, we obtain the PDE

$$\begin{cases} 0 = r + Łu - \alpha u & x \in C \\ u = 0 & x \in \partial C \end{cases}.$$

6.2. How might you use part 6.1 to construct a numerical method that converges to

$$\mathrm{E}_x\left[\int_0^\infty e^{-\alpha t} r(X_t) dt\right]? \tag{1}$$

(Note that the ODE for (1) does not have any naturally associated boundary conditions. Without such boundary conditions on the ODE, (1) may have multiple solutions, making it challenging to numerically compute the correct solution for (1) directly.) **(2 points)**

Solution: Heuristicaly, we expect that as

$$\lim_{b\to\infty} T \to \infty.$$

From which we can see that the solution need only be zero at infinity. One possible solution (which requires technical justification which we will ignore for this class) is that

$$\lim_{b\to\infty} u(x) \to u_\infty(x).$$

Computationally we can interatively increase the value of b and try to use the previous solution as a initial value for the current solution. Without verification, we do not know if this converges either, but we hope it does.

6.3. Suppose that we wish to compute

$$\mathrm{E}_x\left[\int_0^t e^{-\alpha s} r(X_s)ds\right]$$

What partial differential equation (PDE) would you solve to compute this expectation? Include any boundary conditions. (Hint: Note that we are asking for a PDE, not an ODE.) **(5 points)**

Solution: Here we follow a similar procedure as in the first part, only the expectation depends of t as well.

$$u(t,x) = \mathrm{E}_x\left[\int_0^t e^{-\alpha s} r(X_s)ds\right]$$

Using iterated expectation we have

$$\begin{aligned}
&= \mathrm{E}_x\left[\mathrm{E}\left[\int_0^t e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right]\right] \\
&= \mathrm{E}_x\left[\mathrm{E}\left[\int_0^h e^{-\alpha s} r(X_s)ds + \int_h^t e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right]\right]
\end{aligned}$$

Using the Taylor expansion for the first expectation we have

$$= hr(x) + \mathrm{E}_x\left[\mathrm{E}\left[\int_h^t e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right]\right]$$

Using the Markov property and the Taylor approximation for the discount factor this becomes

$$= hr(x) + (1-\alpha h)\mathrm{E}_x\left[\mathrm{E}\left[\int_0^{t-h} e^{-\alpha u} r(X_{h+u})du | X(h)\right]\right]$$

Again we see that we have the same path functional u which gives

$$= hr(x) + (1-\alpha h)\mathrm{E}_x\left[u(t-h, X(h))\right]$$

Subtracting u from both sides, dividing by h and sending $h \downarrow 0$ we obtain the PDE

$$\begin{cases} u_t = r - \alpha u + Łu & t > 0 \\ u(x,0) = 0 & t = 0 \end{cases}.$$

6.4. Suppose that you wish to compute

$$\mathrm{E}_x\left[\int_0^{\min\{T,t\}} e^{-\alpha s} r(X_s)ds\right]$$

What PDE would you solve to compute this expectation? Include any boundary conditions. **(4 points)**

Solution: Here is some quick intuition for this problem: you run the process X_t collecting instantanious dicounted reward $e^{\alpha t}r(X_t)dt$ at time t, until you hit the boundary, at which point you cease to earn any reward. The heuristic derivation of this follows:

Note: $\min\{t, T\} = t \wedge T$.

$$u(t,x) = \mathrm{E}_x\left[\int_0^{t\wedge T} e^{-\alpha s} r(X_s)ds\right]$$

we again use iterated expectation and split the integeral for some $h < t$, :

$$= \mathrm{E}_x\left[\mathrm{E}\left[\int_0^{h\wedge T} e^{-\alpha s} r(X_s)ds + \int_{h\wedge T}^{t\wedge T} e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right]\right]$$

if h small $h \wedge T$ is small and so we can approximate the first term as

$$= \mathrm{E}_x\left[(h\wedge T)r(x) + \mathrm{E}\left[\int_{h\wedge T}^{t\wedge T} e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right]\right].$$

We now focus on the second term. Note that the value of $h \wedge T$ can be determined from the given information of the proces $X(u) : 0 \leq u \leq h$, so this suggests writing

$$\begin{aligned}\mathrm{E}\left[\int_{h\wedge T}^{t\wedge T} e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right] &= I(h > T)\mathrm{E}\left[\int_{h\wedge T}^{t\wedge T} e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h, h > T\right] \\ &\quad + I(h < T)\mathrm{E}\left[\int_{h\wedge T}^{t\wedge T} e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h, T < h\right]\end{aligned}$$

which simplifies by noting that when $h > T$, $h \wedge T = t \wedge T = T$, and when $h < T$, $h \wedge T = h$, which gives

$$= I(h > T) \cdot 0 + I(h < T)\mathrm{E}\left[\int_{h}^{t\wedge T} e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right]$$

Recombining this with the above we have that

$$u(t,x) = \mathrm{E}_x\left[I(h > T)(Tr(x)) + I(h < T)\left(hr(x) + \mathrm{E}\left[\int_{h}^{t\wedge T} e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right]\right)\right].$$

Observe now that (at least heuristically) as $h \downarrow 0$ the indicators become functions of the state x and that $I(h > T) \to I(x \in \partial C)$ (and $T \to 0$) and that $I(h < T) \to T(x \in C)$. Looking at this, we see that we arrive at two different equations, depending on which state we are in:

$$u(t,x) = \mathrm{E}_x\left[I(x \in \partial C) \cdot 0 + I(x \in C)\left(hr(x) + \mathrm{E}\left[\int_{h}^{t\wedge T} e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right]\right)\right]$$

Re-writing this we hat

$$\begin{cases} u(t,x) = 0 & x \in \partial C \\ u(t,x) = hr(x) + \mathrm{E}\left[\int_{h}^{t\wedge T} e^{-\alpha s} r(X_s)ds | X(u) : 0 \leq u \leq h\right] & x \in C \end{cases}$$

Analysis of the second equation is the same as in part 6.3. We can interpret the first equation as a boundary condition to the PDE and thus, the associated PDE is

$$\begin{cases} u_t = r + \mathrm{L}u - \alpha u & x \in C, t > 0 \\ u(x,t) = 0 & x \in \partial C \\ u(x,0) = 0 & t = 0 \end{cases}$$

6.5. How would you change your solution to 6.3 if X instead satisfies

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t?$$

(5 points)

Solution: What complicates matters here is that the coefficients are no longer time homogeneous and we can no longer immediately say that the expectation starting at $X(h)$ at time h and running for $t-h$ units is the same as starting at $X(h)$ at time 0 and running for $t-h$ time units (as the coefficients of the SDE now change in time). We develop a *backwards equation* to overcome this problem.

Let

$$u(x, s; t) = \mathrm{E}_{x,s}\left[\int_s^t e^{-\alpha s} r(X_s)ds\right]$$

and note that $u(x, 0; t)$ is the quantity we are asked to compute. Using iterated expectation and spliting the integral we have

$$\begin{aligned} &= \mathrm{E}_{x,s}\left[\mathrm{E}\left[\int_s^{s+h} e^{-\alpha w} r(X_w)dw + \int_{s+h}^t e^{-\alpha w} r(X_w)dw | X(u) : s \le u \le s+h\right]\right] \\ &= e^{-\alpha s} r(x)h + \mathrm{E}_{x,s}\left[\mathrm{E}\left[\int_{s+h}^t e^{-\alpha w} r(X_w)dw | X(s+h)\right]\right] \end{aligned}$$

Note that we now have the same path functional as above, so this becomes

$$= e^{-\alpha s} r(x)h + \mathrm{E}_{x,s}\left[u(X(s+h), s+h; t)\right]$$

We now apply the standard tricks to obtain a PDE

$$\begin{cases} 0 = u_t + \mathrm{L}^s u - \alpha u + r & s < t \\ u(x,t,t) = 0 & s = t \end{cases}.$$

where

$$L^s = \mu(s,x)\frac{\partial}{\partial x} + \frac{\sigma^2(s,x)}{2}\frac{\partial^2}{\partial x^2}.$$

This is a backwards equation for $u(x, 0)$.