CME308 Final Exam

3:30 PM

June 12, 2007

This exam is open notes and open book. You have three hours. There are a total of 80 points. Good luck!

Problem 1 (15 points): Consider a single-server queue in which customers are served according to a last in / first out non-preemptive discipline. When a given customer completes service, the server begins processing the most recently arrived customer. Customers arrive at times 1,3,5,7,11 and 15. With corresponding service requirements 4,3,1,8 and 2.

1.1. How many customers are in the system at $t = 14.7$? **(7 points)**

1.2. Suppose we run the model 6 times and there are 3, 4, 4, 5, 2 and 1 customers in the queue at time 14.7 for each run. Produce a 90% confidence interval for the expected number of customers in the system at time 14.7. **(4 points)**

1.3. Give an algorithm for generating random variables from the distribution function

$$F(z) = \begin{cases} 1 - \exp(-\lambda(z-1)^\alpha) & z \geq 1 \\ 0 & \text{o.w.} \end{cases}$$

(4 points)

Problem 2 (10 points): Suppose that we are designing staffing levels for a marketing firm that uses home-based part-time employees to market their products over the phone. Each day, a random number, N, of employees will make time available to market the company's products, earning the company Y dollars in revenue. The company wishes to compute the average revenue per employee-day, denoted α. Assume that the sequence of random vectors $((Y_i, N_i) : i \geq 1)$ is i.i.d., where Y_i is the revenue on day i and N_i is the number of working employees on day i.

2.1. The quantity α is the limit, as $n \to \infty$, of

$$\frac{Y_1 + \cdots + Y_n}{N_1 + \cdots + N_n}.$$

Compute α in terms of the moments of Y and N. **(2 points)**

2.2. If n is large, give a complete description of how you would compute a 95% confidence intervale for α based on $(Y_1, N_1), \cdots, (Y_n, N_n)$. **(8 points)**

Problem 3 (10 points): In numerically simulating solutions $X = (X_t : t \geq 0)$ to stochastic differential equations, both first and second order schemes are available. If a k^{th} order scheme is used ($k = 1, 2$), then an approximation X_h is simulated (involving simulating a discrete time approximation $X_h(0), X_h(h), X_h(2h), \cdots, X_h(nh), \cdots$) having a bias that satisfies

$$\mathrm{E}\left[f(X_h(1))\right] = \mathrm{E}\left[f(X(1))\right] + b_k h^k + \mathrm{o}(h^k)$$

as $h \downarrow 0$ for f smooth (and for some constant b_k).

3.1. Suppose that we simulate n i.i.d. copies of the process X_h to time 1, thereby yielding $X_h^1(1), \cdots, X_h^n(1)$. We obtain an estimate

$$\frac{1}{n}\sum_{i=1}^{n} f(X_h^i(1))$$

for $\mathrm{E}\left[f(X(1))\right]$. What is the mean square error of this estimator for n large and h small? **(4 points)**

3.2. The total computation time required to simulate n such copies is roughly of order $c = n/h$. Subject to $n/h = c$,, what is the optimal allocation of n and h? What is the associated root mean square error convergence rate for a k^{th} order scheme under the optimal selection of n and h? **(6 points)**

Problem 4 (14 points): Suppose that we run a newsstand that sells the *New York Times*. Assume that the demand on day i is Poisson distributed with unknown parameter λ^* and i.i.d. over time. (If your solution to a given part involves an optimization problem, just formulate it. Do not solve!)

4.1. Assume you observe the full demand over 5 days: 300, 200, 700, 400 and 800. What is your estimate $\hat{\lambda}$ for λ^*? **(4 points)**

4.2. Suppose that you order 500 newspapers each day. You sell 400, 300, 500, 500 and 500 newspapers over 5 days and do not observe the unmet demand. How would you now estimate λ^*? **(5 points)**

4.3. (continuation of 4.1) Suppose that you believe that demand is increasing linearly in time, so that the demand on day i is Poisson distributed with parameter $\lambda_i^* = a^* i + b^*$ for unknown a^* and b^*. The demand over 5 days is 300, 500, 400, 800 and 900 newspapers respectively. How would you estimate a^* and b^*? **(5 points)**

Problem 5 (11 points): A spider hunting a fly moves between locations 1 and 2 according to a Markov chain with transition matrix P_s starting in location 1. The fly, unaware of the spider, starts in location 2 and moves according to a Markov chain with transition matrix P_f. The spider catches the fly and the hunt ends whenever they meet in the same locations.

$$P_s = \begin{bmatrix} 07 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} \qquad\qquad P_f = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$$

Show that the progress of the hunt, except for knowing the location where it ends, can be described by a three-state Markov chain where one absorbing state represents hunt ended and the other two that the spider and fly are at different locations.

5.1. Obtain the transition matrix for this chain. **(4 points)**

5.2. Find the probability that at time n the spider and fly are both at their initial locations. **(3 points)**

5.3. What is the average duration of the hunt. **(4 points)**

If you are short on time please specify the equations you would solve in each case before computing them.

Problem 6 (20 points): Let $X = \{X_t\}_{t\geq 0}$ be the solution to a stochastic differential equation of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t.$$

Let $\alpha > 0$ and $r : \mathbb{R} \to [0, \infty)$.

6.1. Suppose that we wish to compute

$$\mathrm{E}_x\left[\int_0^T \exp^{-\alpha t} r(X_t)dt\right]$$

where $T = \inf\{t \geq 0 : |X_t - x| \geq b\}$. What ODE would you solve? (Include any necessary boundary conditions.) **(4 points)**

6.2. How might you use part 6.1 to construct a numerical method that converges to

$$\mathrm{E}_x\left[\int_0^\infty e^{-\alpha t} r(X_t)dt\right]? \tag{1}$$

(Note that the ODE for (1) does not have any naturally associated boundary conditions. Without such boundary conditions on the ODE, (1) may have multiple solutions, making it challenging to numerically compute the correct solution for (1) directly.) **(2 points)**

6.3. Suppose that we wish to compute

$$\mathrm{E}_x\left[\int_0^t e^{-\alpha s} r(X_s)ds\right]$$

What partial differential equation (PDE) would you solve to compute this expectation? Include any boundary conditions. (Hint: Note that we are asking for a PDE, not an ODE.) **(5 points)**

6.4. Suppose that you wish to compute

$$\mathrm{E}_x\left[\int_0^{\min\{T,t\}} e^{-\alpha s} r(X_s)ds\right]$$

What PDE would you solve to compute this expectation? Include any boundary conditions. **(4 points)**

6.5. How would you change your solution to 6.3 if X instead satisfies

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t?$$

(5 points)