CME 305: Discrete Mathematics and Algorithms

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Midterm Solutions

Problem 1. We say that a graph is k-regular if all its vertices have degree k. Prove that all k-regular bipartite graphs have a perfect matching.

Hint: use Hall's Marriage Theorem

Solution: Given $G(A, B, E)$ select a subset $S \in A$. The number of edges leaving this set is $k|S|$ (since the graph is k-regular) which equals to the sum of the degrees of vertices in S. Since each of these edges must enter $N(S)$, the neighborhood of S, we have,

$$k|S| = d(S) = \sum_{v \in S} d(v) \leq \sum_{u \in N(S)} d(u) = d(N(S)) = k|N(S)|$$

For $S = A$ and $k \geq 1$, $N(A) = B$, the inequality above is satisfied exactly and we conclude that $|A| = |B|$. Furthermore we have for any subset $|S| \leq |N(S)|$. Thus by Hall's Marriage theorem G has a perfect matching.

Problem 2. A proper coloring of the edges of a graph G is an assignment of colors to edges such that no two adjacent edges have the same color. A graph is said to be k-edge-colorable if there is a proper edge-coloring of the graph that uses k or fewer colors.

Prove that if the maximum degree in a bipartite graph G is Δ, then G is Δ-edge-colorable. Also provide a polynomial-time algorithm that produces such a coloring.

Solution: We construct a graph G' from $G(A, B, E)$ as follows. Add enough vertices to A or B so that $n = |A| = |B|$. Since the graph is bipartite we have $r = d(A) = d(B) = |E|$. Perform the following procedure. While $r < \Delta n$, add egde (u, v) for all $u \in A$ and $v \in B$ such that $d(u) < \Delta$ and $d(v) < \Delta$ incrementing r by one after each step.

At each step $d(A) = d(B) < \Delta n$ and $d(w) \leq \Delta$ for all vertices w. Thus the pair (u, v) exists. Adding an edge inreases both $d(A)$ and $d(B)$ by one. So after some finite number of steps we have $r = \Delta n$ and $d(w) \leq \Delta$ which implies the constucted graph G' is Δ-regular.

From problem 1 we know that G' has a perfect matching M'. Color these edges (no two are adjacent) with color c_0 and remove them from the graph obtaining a new graph G''. Note that G'' is $(\Delta - 1)$-regular and again has a perfect matching M'' which we can color with c_1. When all edges are gone we've used Δ colors and no two adjacent edges have the same color. Thus G' is proper Δ-edge-colorable.

Finally, since removing edges and vertices from a graph does not damage a proper edge coloring, G, a subgraph of G', is also Δ-edge-colorable.

Problem 3. (extra credit) Give a polynomial time algorithm for a proper coloring of the vertices of a 3-vertex-colorable graph of size n with at most $O(\sqrt{n})$ colors.

Solution: See solutions to Homework 3 (extra credit).