CME 305: Discrete Mathematics and Algorithms

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Lecture 7: More Network Flow Applications

Chapter 7 in Kleiberg and Tardos [KT] discusses many applications of network flow to solve different types of problems. These include Project Scheduling, Baseball Elimination and Airline Flight Scheduling covered in this lecture.

Scheduling – Project Selection

Problem: We are given:

- a set of projects $T_1, \ldots, T_n$,
- project T_i has a profit P_i which can be positive or negative,
- a project can be a prerequisite of another.

Goal: choose a subset of projects that satisfy the prerequisite requirement and have the maximum total profit.

Solution: For every project j with positive profit, put an edge from T_j to the target node t with capacity P_j. For each project i with negative profit, put an edge from the source node s to T_i. For each prerequiste, put an edge with capacity infinity. Find the minimum s-t cut (A, B) in this network.

Claim 1 *Jobs in B have all their prerequisites met.*

Proof: Suppose not. Then there is an edge with capacity ∞ going from A to B. This cannot be, since (A, B) is a min-cut. ■

Claim 2 *B is an optimal job-set.*

Proof: Let I be the total possible surpluses, i.e. the total weight of edges going into t. Then total amount of profit for job-set B is equal to $I - c(A, B)$, because edges in the cut either represent costs or potential surplus jobs. Therefore, minimizing the cut maximizes profits. ■

Baseball Elimination

Problem: Suppose we have a set of teams $T_1, \ldots, T_j$ each with a win record $W_1, \ldots, T_j$, and a schedule of k games remaining in the season $G_1, \ldots, G_k$, where each G_i is an unordered pair of teams. The question is, does there exist a scenario under which team T_j ends the season with at least as many wins as any other team? In other words, has T_j been eliminated from first place?

Solution: We will represent this as a flow problem. First, we can assume that T_j wins its remaining p games, so credit T_j with these wins and remove those games from the schedule. Next, for each team $i \neq j$ put a vertex and attach a directed edge to the sink with capacity equal to the maximum number of games it can win and not exceed T_j's maximum win total. For each game G_i, put an edge from the source with capacity 1, and put an edge to each participating team with capacity 1.

Claim 3 *T_j has not been eliminated if and only if the preceding graph has a flow with value $k - p$.*

Proof: "$\Leftarrow$" Since each game has capacity 1, the integral flow theorem a flow implies that a flow of value $k - p$ assigns each game to a team without exceeding their maximum win totals.

"$\Rightarrow$" If T_j has not been eliminated, there exists an assignment of games to teams that does not exceed their maximum win totals. This defines a flow of value $k - p$.

■

Scheduling – Airline Flights

This problem modifies the original network flow problem by introducing lower bounds on flow in addition to capacities on the edges. To see a treatment of this and other extensions as well as how to solve this problem see Section 7.7 in Kleinberg and Tardos [KT].