Instructor: Professor Amin Saberi (saberi@stanford.edu)
January 27, 2009

Lecture 6: More Network Flow and Applications

Recall from last lecture the *Ford-Fulkerson* algorithm for max-flow:

Algorithm 1 Ford-Fulkerson, 1956

```
Start with f(e) = 0, ∀e ∈ E.
while there is a path P from s to t in R(N, f) do
    send a flow of value min_{e∈P} c(e) in R along P.
    augment f in N using the above flow.
    rebuild the residual network R(N, f).
end while
Output f.
```

We showed that,

Lemma 1 *If Ford-Fulkerson terminates, it outputs a maximum flow.*

We proved this by noting that when Ford-Fulkerson terminates, there are no s-t paths in the residual network R. If we take all nodes reachable in R from s and call this node set A, this defines an s-t cut $(A, V/A)$ across which all capacities are saturated.

Lemma 2 *If all edge capacities of N are integral, Ford-Fulkerson terminates.*

Proof: At each stage in Ford-Fulkerson, we construct the residual network R. If there exists an s-t path in R, then we increase flow by at least 1 (since capacities are integral). Since edge capacities are finite, the algorithm must terminate in finite time. ∎

Note that although this proves that Ford-Fulkerson terminates in a *finite* number of steps, its running time may depend on the maximum value of the capacities, since it is only guaranteed to increment by one at each step.

Two important corollaries follow from the proof of Ford-Fulkerson:

Corollary 1 (Max-Flow/Min-Cut) *The minimum cut value in a network is the same as the maximum flow value.*

Corollary 2 (Integral Flow) *If all edge capacities in a network are integer, then there exists an integral maximum flow.*

Not only does Ford-Fulkerson prove the existence of a min-cut of value equal to the max-flow, it gives a mechanism for finding it, as the cut $(A, V/A)$ referenced in the proof is a min-cut.

The importance of flow integrality becomes apparent in the following application.

Application: Maximum matching in a bipartite graph.

Recall the maximum matching problem defined in lecture 4. A maximum matching $M^* \subseteq E$ is a set of edges satisfying two criteria:

- No two edges in M^* share an endpoint.
- $|M^*|$ is the maximum edge set satisfying condition 1.

We will represent this as a flow problem. We create a source node s and add edges from s to each node in A, and create a target node t and add edges from each node in B to t. We put capacity 1 on each edge. Then any matching defines a flow in this graph, and any integral flow defines a matching (consider the saturated edges in between A and B). Therefore, finding maximum integral flow in this graph yields the maximum matching. By corollary 2, we know that such a flow exists and that Ford-Fulkerson will find it.

Lemma 3 *Hall's marriage theorem is implied by max-flow/min-cut.*

Proof: Define the flow problem as previously with the exception of placing a capacity of $c(e) = n + 1$ for each $e \in E$ (all edges except those leaving s or entering t). Let $cut(R, Q)$ be the min s-t cut found by solving the max-flow problem. Define $A_R = A \cap R$, $B_R = B \cap R$ and $A_Q = A \cap Q$, $B_Q = B \cap Q$.

First, note that if the max-flow has a value $v(f) = n$, we have found a perfect matching as shown previously. Then clearly no subset $S \subset A$ has a neighborhood $N(S)$ for which $|N(S)| < |S|$ (otherwise a perfect matching would not be possible) in agreement with Hall's marriage theorem.

Now, note that there cannot be any edges between A_R and B_Q. If such an edge e existed, it would have to be in the $cut(R, Q)$ by definition and would add a value of $c(e) = n + 1$. This contradicts the fact that we have min cut since the maximum flow possible out of the source s is n by construction. Thus (1) $N(A_R) \subseteq B_R$. Furthermore, we can conclude that all flow goes along edges from s to A_Q and from B_R to t. Thus, since all edges in the min-cut are saturated, (2) $v(f) = |A_Q| + |B_R|$.

Finally, consider a scenario where the max-flow has a value $v(f) < n$. By (1) we have $|N(A_R)| \leq |B_R|$ and by (2) we have $|B_R| = v(f) - |A_Q| < n - |A_Q| = |A_R|$. We can now conclude that $|A_R| > |N(A_R)|$. Thus we have found a subset whose neighborhood is strictly larger and by Hall's marriage theorem a perfect matching does not exist. ■

Application: Counting edge-disjoint paths

Let $G(V, E)$ be a graph and let $P_1, P_2, \ldots P_k$ be a collection of paths from s to t. We say that $P_1, P_2, \ldots P_k$ are **edge disjoint** if they have no edges in common.

Say that we wish to count the maximum number of edge disjoint paths between s and t. Then we simply set capacities of all edges in G to be 1. As another application of the integral max-flow corollary, it is easy to see that the value of the max-flow in the flow network will be the number of edge disjoint paths.

The max-flow/min-cut theorem therefore gives an easy proof of the relationship between the number of edges we need to remove to disconnect s and t and the number of edge-disjoint paths known as *Menger's Theorem*:

Theorem 1 (Menger, 1927) *The minimum number of edges necessary to disconnect two given vertices s and t is equal to the maximum number of edge-disjoint paths from s to t.*

We call a graph $G(V, E)$ *k-connected* if it takes the removal of at least k edges to disconnect G. A corollary of Menger's theorem is then

Corollary 3 *G is k-connected if and only if there are at least k edge-disjoint paths between every pair of vertices.*