CME 305: Discrete Mathematics and Algorithms
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Lecture 16: Marriage, Stability and Honesty

Assume that we have n men and m women, and that each person gives a ranking in order of preference of members of the opposite sex. The *stable marriage* problem asks to pair the men and women in such a way that no two persons prefer each other over their partner.

Example: Assume that we have men labeled a, b, c, and women labeled $1, 2, 3$ with preferences (rankings) given by,

$$\begin{array}{llll} a: & 132 & 1: & abc \\ b: & 231 & 2: & bca \\ c: & 123 & 3: & acb \end{array}$$

A **matching** is a one-to-one correspondence, e.g.

$$1 \leftrightarrow b \quad 2 \leftrightarrow a \quad 3 \leftrightarrow c.$$

For a matching M, a pair (m, w) is a **blocking pair** if and only iff both m and w prefer each other to their current partners $P_M(m)$ and $P_M(w)$. A matching is **stable** if it doesn't have a blocking pair. In the example above,

$$1 \leftrightarrow a \quad 2 \leftrightarrow b \quad 3 \leftrightarrow c$$

is a stable matching.

How can we find a stable matching? Gale and Shapley, in 1962, proposed the Defered Acceptance Algorithm 1.

Claim 1 *The GS algorithm terminates.*

Proof: A man is rejected at most m times, each time removing a woman from his preference list. Each list is of size m, thus we take at most nm steps. ∎

Claim 2 *GS algorithm produces a perfect matching whenever $m = n$.*

Proof: Since $m = n$ for every unmatched man the must be an unmatched woman. Suppose there is some unmatched man m and unmatched woman w. Since w is unmatched she has never rejected a proposal. Since m is unmatched he has always been rejected. This is a contradiction since w is in some position in m's preference list. ∎

Algorithm 1 Defered Acceptance [Gale-Shapley] (men-proposing version)

let all men and women be unmatched.
repeat
  unmatched man m proposes to most prefered woman w in his list
  if w is unmatched **then**
    match w and m
  else if w is matched to m' and she prefers m to m' **then**
    match w and m and leave m' unmatched.
  else
    m removes w from his preference list
  end if
until every man or every woman is matched
return matching M of all pairs $(m, P_M(m))$

Claim 3 *The matching found by the GS algorithm is stable.*

Proof: Suppose (m, w) is a blocking pair for the result of GS algorithm M. If m has proposed to w and was rejected, it means w prefers her current partner to m. If m has not proposed to w, then m prefers his current partner to w. So there cannot be a blocking pair. ■

Theorem 1 *The men-proposing algorithm is* man-optimal *in a very strong sense: every man will be matched to the* best *partner he could be matched to in any stable matching.*

Proof: By contradiction.
Denote M as the matching produced by Algorithm 1. Consider the **first** event that a man is rejected by a woman in the algorithm. Label this man m, the woman w and suppose there exists some other stable matching M' such that m and w are matched i.e. $m = P_{M'}(w)$.

Let $m' = P_M(w)$, and $w' = P_{M'}(m')$. Then we have that w must prefer m' to m, otherwise w wouldn't have rejected m over m'.

Now, if m' prefers w to w', then M' is not stable, contradiction.

And, if m' prefers w' to w, it means m' was rejected by w' since m' and w are matched in M, contradicting that m getting rejected was the first such event. ■

Theorem 2 *The men-proposing algorithm is* female-pessimal*: every woman will be matched to the* worst *partner she could be matched to in any stable matching.*

Proof: Again by contradiction, let M be the output of Algorithm 1. Suppose there is another stable matching M' and w prefers $m' = P_{M'}(w)$ to $m = P_M(w)$. Let $w' = P_{M'}(m)$. By man-optimality, m prefers w to w'. Therefore, (m, w) is a blocking pair for M'. ■

Extensions and Applications

Generalization to stable marriage present some difficulties. In more realistic scenarios we can expects that ranking lists to be incomplete, a rather be alone than with someone preference, and to model indifference between subsets of partners. While each of these are solvable in polynomial time separately, solving incomplete lists with indifference preference together makes the problem NP-hard.

Another issue of concern is whether there are any incentives to change preference lists. In game theory, a game is called *strategyproof (thruthful)*, if players have no incentive to hide information from each other. Could someone state false preferences to gain a benefit from the match? For men, it is trivially truthful, but there is no incentive for women to be truthful! One can design cases where changing the list of preferences will increase the happiness of a given woman with the final match results.

Furthermore, in the real world matching with preferences is usually not one to one. Many to one marriage problems, commonly referred to as the college admissions model is famously employed in several entry level professional labor markets such as the National Residency Matching Program (NRMP). The college admission model assumes some population of students, a number of colleges with certain student capacities and preferences stated by the colleges and the students.

NRMP has been in effect since the 1950's and has experimented with different matching models to match medical school graduates to hospitals as they enter residency programs. For a long time the system was run as *hospital optimal*. For this reason, the system was sued for being anti-competitive; the system held up, but was switched from being hospital optimal to resident optimal. Interestingly, this switch didn't change the rankings much, because rankings are highly correlated.

Rules for the algorithm used to conduct the matching for NRMP have been updated many times through the years as complications arose. One of the major ones was the introduction of couples, that is an application to a hospital coming not from a single student but from a married couple. Economist, Al Roth, proved that introducing couples into the match leaves a possibility that no stable match exists.