CME 305: Discrete Mathematics and Algorithms

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HW#2 – Due 02/17/09

1. Given a nonincreasing sequence of positive integers $d_1, d_2, \ldots, d_n$, suppose we wish to construct a simple graph $G(V, E)$ on n vertices where d_i is the degree of $v_i \in V$.

   (a) Prove that there exists a simple graph with sequence $d_1, d_2, \ldots, d_n$ if and only if there exist one with sequence $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n$. Use this to design a greedy polynomial-time algorithm to construct G or determine that such a graph cannot exist.

   (b) Now assume the sequence given above is partitioned as $a = d_1, d_2, \ldots, d_k$ and $b = d_{k+1}, d_{k+2}, \ldots, d_n$ for $1 \leq k < n$. Show how to use network flow to construct a simple bipartite graph $G(A, B, E)$ (or determine that one does not exist) such that the sequences a and b describe the degrees of vertices in A and B respectively.

2. Given a network N with integral capacities, suppose we wish to identify, from among all minimum cuts, a minimum cut containing the least number of edges. Show how to modify the capacities in N in order to find the cut with the least number of edges.

3. Kleinberg and Tardos Problem 7.27

4. Kleinberg and Tardos Problem 7.31

5. Recall that to show a problem X is NP-complete we must show that it is in NP and construct a polynomial-time computable function that maps inputs of a known NP-complete problem to inputs of X (e.g. 3-SAT $\leq_p X$) in a way that preserves problem satisfiability. The last statement would imply that all problems in NP are polynomial reducible to X, hence X is NP-complete.

   (a) Integer Programming (IP) decision problem asks whether there exists an integer solution x satisfying linear constraints $Ax \leq b$ and with objective value $c^T x$ at least k. Prove that IP is NP-complete.

   (b) The Clique decision problem asks whether a clique (a complete subgraph) of size k exists in given graph G. Prove that Clique is NP-complete.

   (c) Consider a modified version of Clique in which all vertices have degree at most 3. Is this problem NP-complete? Why or why not?

   (Hint: Use 3-SAT for above reductions, or any other NP-Complete problem from class)

6. For the selected project submit an (upto) one page write-up describing what the central problem is including some history, branches of mathematics involved, why the problem is difficult, and what the applications are. Give links to a couple of relevant papers and summarize each in a few sentences. It is okay to use material from the Project pages but hopefully you can expand a bit on what is there now. Finally list some ideas you and your group have on what you can accomplish during the remainder of the quarter. Submit only one write-up per group.