CME 305: Discrete Mathematics and Algorithms

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HW#1 – Due 02/03/09

1. For any undirected graph $G(V, E)$, calculate the quantity $\frac{1}{|E|}\sum_{v_i \in V} d(v_i)$ where the degree of vertex v_i is given by $d(v_i)$. From this deduce that in any graph the number of vertices of odd degree is even.

2. Recall the definition of a bipartite graph. Let $G(V, E)$ be a graph and (A, B) be a partition of V. We say that G is bipartite if all edges in E have one end-point in A and the other in B. More precisely, for all $(u, v) \in E$ either $u \in A, v \in B$ or $u \in B, v \in A$.

   (a) Prove that a graph is bipartite if and only if it doesn't have an odd cycle.

   (b) A graph is called k-regular if all vertices have degree k. Prove that if a bipartite G is also k-regular with $k \geq 1$ then $|A| = |B|$.

3. Let $G(V, E)$ be a simple graph. Define its complement $\bar{G}$ as a graph on the vertex set V with an edge set $\bar{E}$ (the complement of E).

   (a) What is the degree sequence of $\bar{G}$ in terms of the degree sequence of G.

   (b) An automorphism of a graph G is a permutation of its vertices which preserves adjacency (i.e. $(u, v) \in E \Leftrightarrow (\phi(u), \phi(v)) \in E$). Let $Aut(G)$ be a set of automorphisms of G. Show that $Aut(G) = Aut(\bar{G})$.

   (c) Prove that at least one of G and $\bar{G}$ is connected.

4. Prove Cayley's theorem by linear algebra. Recall that Cayley's theorem states that the number of labeled trees on n vertices is n^{n-2}.

   (a) An oriented incidence matrix B of a directed graph $G(V, E)$ is a matrix with $n = |V|$ rows and $m = |E|$ columns with entry B_{ve} equal to 1 if edge e enters vertex v and -1 if it leaves vertex v. Let $M = BB^T$. Show that for any $i \in \{1, \ldots, n\}$,

   $$\det M_{ii} = \sum_N (\det N)^2,$$

   where $M_{ii} = M \backslash \{i^{th} \text{ row and column}\}$, and N runs over all $(n-1) \times (n-1)$ submatrices of $B \backslash \{i^{th} \text{ row}\}$. Note that each submatrix N corresponds to a choice of $n - 1$ edges of G.

   (b) Show that

   $$\det N = \begin{cases} \pm 1 & \text{if edges form a tree} \\ 0 & \text{otherwise} \end{cases}$$

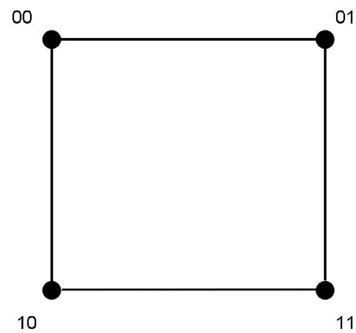
   This implies that $t(G) = \det M_{ii}$, where $t(G)$ is the number of spanning trees of G. In this definition of a tree, we treat a directed edge as an undirected one.

   (c) Show that for the complete graph on n vertices K_n,

   $$\det M_{ii} = n^{n-2}.$$

   Conclude that this implies Cayley theorem.

5. An n-dimensional cube can be represented by a graph with 2^n vertices with every vertex corresponding to an n-bit binary number. Two vertices are connected by an edge if their corresponding binary numbers differ by only one bit. For example, the following represents a 2-D cube.



Prove that every n-dimensional cube has a Hamiltonian cycle.

6. A balanced digraph $G(V, E)$ is a directed graph with the in-degree and the out-degree equal for each vertex. An tree in a directed graph is a set of edges such that if directions were ingored the resulting graph is a tree. An in-tree is a (directed) tree with a root vertex v such that from any vertex u we can follow a directed path to v. An out-tree (arborescence) is one where the directed paths are from the root to the other vertices.

   Consider the following algorithm: Given a connected balanced digraph $G(V, E)$, choose a spanning in-tree T rooted at any vertex $v \in V$. Start a path from v and traverse the edges of G. The rule for choosing the next edge at every vertex u is that if there is an unvisited edge going out of u that does not belong to T, traverse that edge. Otherwise, take the tree-edge going out of u. Continue the path until there is no way out.

   (a) Give a reason why you can always choose the spanning in-tree T.

   (b) Prove that when the above algorithm stops every edge is traversed exactly once.

   (c) Prove that the number of Eulerian circuits in G is given by

   $$d(G) \prod_{v \in V} (outdeg(v) - 1)!$$

   where $d(G)$ is the number of spanning arborescences of G rooted at any vertex v.

7. Form a project team of 2 to 4 people and decide on a topic. Submit a list of partners and a project topic. Topic ideas are available on the webpage; you may choose another if you like, but should clear it with Amin. If you need help finding a team let us know.

   The final expectations for the project are (1) a literature survey, (2) understanding of chosen topic and an exposition of current state of the art (3) some type of extension e.g. coding up an algorithm, proving a lemma missing from the paper, solving an open problem etc...