

# Equivalence of norms for finite dimensional spaces

**Lemma 1.** *If  $a_1, \dots, a_n \in \mathbb{C}$  then*

$$\frac{1}{\sqrt{n}} \sqrt{\sum |a_i|^2} \leq \sum |a_i| \leq n \sqrt{\sum |a_i|^2}$$

*Proof.* ☺ □

**Lemma 2.** *Let  $V$  be a normed vector space with basis  $v_1, \dots, v_n$ . Let  $T : \mathbb{C}^n \rightarrow V$  be defined by  $T(e_i) = v_i$ . (Here  $e_1, \dots, e_n$  is the standard basis of  $\mathbb{C}^n$ ). Then  $T$  is continuous.*

*Proof.* For clarity we use  $\|\cdot\|_V$  to denote the norm on  $V$  and  $\|\cdot\|$  to denote the norm on  $\mathbb{C}^n$ . Let  $x = \sum x_i e_i$ ,  $y = \sum y_i e_i$ , and  $M = \max\{\|v_1\|_V, \dots, \|v_n\|_V\}$ . Then

$$\begin{aligned} \|T(x) - T(y)\|_V &= \left\| \sum (x_i - y_i) v_i \right\|_V \\ &\leq \sum |x_i - y_i| \|v_i\|_V \\ &\leq M \sum |x_i - y_i| \\ &\leq nM \sqrt{\sum |x_i - y_i|^2} \\ &= nM \|x - y\| \end{aligned}$$

which immediately implies continuity. □

**Theorem 3.** *Let  $V$  be a finite dimensional normed vector space. Then any two norms on  $V$  are equivalent.*

*Proof.* Let  $v_1, \dots, v_n$  be a basis of  $V$ , and  $\|\cdot\|_V$  be any norm on  $V$ . Let  $T : \mathbb{C}^n \rightarrow V$  be defined as Lemma 2. Note  $T$  is a vector space isomorphism.

We define a new norm on  $V$  by  $\|\sum x_i v_i\|' = \sum |x_i|$  (it is easy to check that  $\|\cdot\|'$  actually defines a norm on  $V$ ). To prove the theorem it is enough to show that  $\|\cdot\|_V$  and  $\|\cdot\|'$  are equivalent.

Let  $x = \sum x_i v_i \in V$ . Then  $\|x\|_V \leq \sum |x_i| \|v_i\|_V \leq M \sum |x_i| \leq Mn \|x\|'$  finishing one half of the proof. For the other half, since  $\{c \in \mathbb{C}^n \mid \|c\| = 1\}$  is compact. Hence the continuous function  $\|Tc\|_V$  attains its infimum, say  $\varepsilon$ , on this set. By non-degeneracy of  $\|\cdot\|_V$  we conclude  $\varepsilon > 0$ . Thus we have for any  $c \in \mathbb{C}^n$ ,

$$(1) \quad \|Tc\|_V = \left\| T \left( \frac{c}{\|c\|} \right) \right\|_V \|c\| \geq \varepsilon \|c\|.$$

Now pick  $x = \sum x_i v_i \in V$ , and put  $c = \sum x_i e_i = T^{-1}x$ . Then

$$\begin{aligned} \|x\|_V &= \|Tc\|_V \\ &\geq \varepsilon \|c\| \\ &= \varepsilon \sqrt{\sum |x_i|^2} \\ &\geq \frac{\varepsilon}{n} \sum |x_i| \\ &= \frac{\varepsilon}{n} \|x\|' \end{aligned} \quad \square$$

**Corollary 4.** *Let  $V$  be a normed vector space,  $U \subseteq V$  a finite dimensional subspace. Then  $U$  is closed.*

*Proof.* Let  $(u_n)$  be a sequence in  $U$ , such that  $(u_n) \rightarrow u$ . To prove the Corollary, it is enough to show  $u \in U$ .

Let  $e_1, \dots, e_d$  be a basis of  $U$ . If  $x = \sum x_i e_i \in U$ , we define  $\|x\|' = \sum |x_i|$ . Note that since  $U$  is finite dimensional, the previous theorem immediately gives that  $\|\cdot\|'$  and  $\|\cdot\|$  are equivalent on  $U$ . Thus  $\exists c > 0$  such that for any  $x \in U$ ,  $\|x\|' \leq c \|x\|$ .

Now let  $u_n = \sum_{i=1}^d a_{i,n} e_i$ . Since  $(u_n) \rightarrow u$ ,  $(u_n)$  is Cauchy, and hence for any  $\varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N} \ni m, n > n_0 \implies \|u_m - u_n\| < \varepsilon$  and hence  $\|u_m - u_n\|' < c\varepsilon$ . This means  $\sum |a_{i,n} - a_{i,m}| < c\varepsilon$ .

The last estimate immediately shows that for any  $i \in \{1, \dots, d\}$  the sequence  $(a_{i,n})_n$  is Cauchy, and hence convergent. Say  $(a_{i,n})_n \rightarrow a_i$ . Hence, using continuity of addition and scalar multiplication (on your homework), we have  $(u_n) \rightarrow \sum a_i e_i \in U$ , concluding the proof. □

**Corollary 5.** *Let  $V$  be a finite dimensional normed vector space. Then  $\overline{B_1(0)} = \{x \in V \mid \|x\| \leq 1\}$  is compact.*

*Proof.* Suppose  $V$  is finite dimensional. Define  $T : \mathbb{C}^d \rightarrow V$  as in Lemma 2. Theorem 3 (or more precisely inequality (1)) shows that  $T$  is a homeomorphism (i.e. both  $T$  and  $T^{-1}$  are continuous). The fact that  $\overline{B_1(0)}$  now follows from the compactness of the unit ball in  $\mathbb{C}^d$ . □