Assignment 8: Spectral theory

Assigned Wed 05/23. Due Wed 05/30.

1. (a) Let $K \in B(X, X)$ be compact Hermitian. Show that $\operatorname{im}(I + K)$ is closed, and $X/\operatorname{im}(I + K)$ is finite dimensional. Further show $\dim\ker(I + K) = \dim(X/\operatorname{im}(I + K))$. [This is still true if K is not Hermitian, but of course false if K is not compact.]

   (b) Let $K \in B(X, X)$ be compact Hermitian. Show that $\sigma(K) - \{0\}$ consists of only eigenvalues. [This is still true if K is not Hermitian, but of course false if K is not compact.]

   (c) Let $K \in B(X, X)$ be compact. Show that if X is infinite dimensional $0 \in \sigma(K)$.

   (d) Let $k : [0, 1] \times [0, 1] \to \mathbb{C}$ be a continuous function such that $k(x, y) = \bar{k}(y, x)$. Then for any $\lambda \in \mathbb{C} - \{0\}$ show that either for all $g \in L^2([0, 1])$ the (functional) equation $\int k(x, y)f(y)dy - \lambda f(x) = g(x)$ has a unique solution, or the (functional) equation $\int k(x, y)f(y) - \lambda f(x) = 0$ has a nonzero solution. [Note that in the equations above, both the left and right hand sides are elements of L^2, and it does not make sense to talk about the values of these functions for any particular x. Thus when we say the (functional) equation $f(x) = g(x)$ holds, we mean that f and g are equal as elements of L^2. This result is true even without the assumption that $k(x, y) = \bar{k}(y, x)$, and is called the *Fredholm Alternative*.]

2. Let $\Omega \subseteq \mathbb{C}$ be open. Recall $f : \Omega \to \mathbb{C}$ is *holomorphic* if for all $z \in \Omega$, $\lim_{\zeta\to z} \frac{f(\zeta)-f(z)}{\zeta-z}$ exists. From standard complex analysis, we recall that holomorphic functions can be represented by power series. Namely for any $z \in \Omega$, there exists a sequence $(c_n)_n$ of complex numbers such that $f(\zeta) = \sum c_n(\zeta-z)^n$, and this series has radius of convergence $\rho = \inf\{|z - z'| \mid z' \notin \Omega\}$.

   (a) Let $A \in B(X, X)$, and $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Show that $f(A) \in B(X, X)$, and $f(\sigma(A)) \subseteq \sigma(f(A))$. [We define $f(A) \in B(X, X)$ as follows: Since f is holomorphic on all of $\mathbb{C}$, we write $f(z) = \sum c_i z^i$ for all $z \in \mathbb{C}$. We know that this series converges absolutely for all $z \in \mathbb{C}$, and thus we define $f(A) \in B(X, X)$ by $f(A) = \sum c_i A^i$.]

   For those familiar with a little more complex analysis, the above can be extended to meromorphic functions.

   (b) Let $f : \mathbb{C} \to \mathbb{C}$ be meromorphic, such that all poles of f are outside $\sigma(A)$. Show that $\sigma(f(A)) = f(\sigma(A))$. [The definition of $f(A)$ is possibly less transparent in this situation: Let $\{\mu_k\}$ be the (discrete) set of poles of f. Then we know that there exists natural numbers n_k, and holomorphic entire functions g, g_k such that $f(z) = g(z) + \sum \frac{1}{(z-\mu_k)^{n_k}} g_k(z)$. Now $g_k(A)$ is defined as in the previous subpart. Now since $\mu_k \notin \sigma(A)$, $\frac{1}{A-\mu_k I}$ can be defined as $(A - \mu_k)^{-1}$. Thus we define $f(A) = g(A) + \sum((A - \mu_k I)^{-1})^{n_k} g_k(A)$.]

   (c) If $n \in \mathbb{Z}$, show that $\sigma(A^n) = \{\lambda^n \mid \lambda \in \sigma(A)\}$. [If $n < 0$, you can assume A is invertible. If you couldn't figure out the last subpart, do this one explicitly. If you got the last subpart, this should be one line.]

3. In the last homework we saw $\triangle^{-1}$ (defined for *periodic* functions) was compact. This is a general principle, valid in a more general setting. We prove this here for functions defined on the unit disk in $\mathbb{C}$, which is generally the first step towards proving it for general domains.

   Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$, and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. We define $C^2(D)$ to be the set of all functions which are twice differentiable on D, and continuous on $\bar{D}$. We say $u \in C^2$ is *harmonic* if $\triangle u = \partial_x^2 u + \partial_y^2 u = 0$. Our first goal is to show that given $f : S^1 \to \mathbb{C}$, find a harmonic function $u : \bar{D} \to \mathbb{C}$ such that $u|_{S^1} = f$.

   (a) Let $n \in \mathbb{N}$. Show that $u(z) = z^n$ and $v(z) = \bar{z}^n$ are Harmonic.

   Given $f : \mathbb{R} \to \mathbb{C}$ periodic, we can identify it with a function $F : S^1 \to \mathbb{C}$, by defining $F(e^{i\theta}) = f(\theta)$. Thus for $F : S^1 \to \mathbb{C}$, we define $\hat{F}(n) = \frac{1}{\sqrt{2\pi}}\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta} = \frac{1}{2\pi i}\oint_{S^1} F(z)z^{-n}\frac{dz}{z}$.

   (b) If $F \in L^2(S^1)$, show that $F(z) = \sum \hat{F}(n)z^n$.

   To extend $F : S^1 \to \mathbb{C}$ to a harmonic function $u : \bar{D} \to \mathbb{C}$, one would be tempted to write $u(z) = \sum \hat{F}(n)z^n$. But unfortunately, this causes problems at the origin when $n < 0$. Thus we realize that when $|z| = 1$, $\bar{z} = z^{-1}$, and hence $F(z) = \hat{F}(0) + \sum_1^\infty(\hat{F}(n)z^n + \hat{F}(-n)\bar{z}^n)$. This expression yields itself to a natural harmonic extension inside D.

(c) Let $F : S^1 \to \mathbb{C}$ be continuous. Define $u : D \to \mathbb{C}$ by $u(z) = \hat{F}(0) + \sum_1^\infty \hat{F}(n) z^n + \hat{F}(-n)\bar{z}^n$. Show that u is harmonic.

(d) When $r < 1$, show that $u(re^{i\theta}) = \int_0^{2\pi} f(\phi) P_r(\theta - \phi)\, d\phi$, where $P_r(\phi) = \frac{1}{2\pi} \frac{1-r^2}{1+r^2-2r\cos\phi}$. P_r is called the Poisson Kernel. We abuse notation and define $f * P : D \to \mathbb{C}$ by $f * P(re^{i\theta}) = u(re^{i\theta}) = \int_0^{2\pi} f(\phi) P_r(\theta - \phi)\, d\phi$.

(e) Show that P_r is an approximate identity. Namely, show that for $r \in [0, 1)$, $\int P_r(\theta)\, d\theta = 1$, $P_r(\theta) \geqslant 0$, and for any $\varepsilon > 0$, $\lim\limits_{r \to 1^-} \int_{|\theta| > \varepsilon} P_r(\theta)\, d\theta = 0$.

(f) Let $F \in L^p(S^1)$. Show that $(u_r) \to F$. Explicitly, show that $\lim\limits_{r \to 1^-} \int \left| u(re^{i\theta}) - F(e^{i\theta}) \right|^p d\theta = 0$.
[Thus you can think of $u : \bar{D} \to \mathbb{C}$ to be a harmonic function such that $u|_{S^1} = F$ in the L^p sense.]

(g) If $F : S^1 \to \mathbb{C}$ is continuous, show that $u : \bar{D} \to \mathbb{C}$ is continuous, (even at the boundary of D), and $u|_{S^1} = F$.

(h) If $\hat{F} \in \ell^\infty$, show that $\triangle u = 0$.

(i) *(Uniqueness)* If $F \in L^p$, and u, v satisfy $\triangle u = \triangle v = 0$ inside D, and $u|_{S^1} = v|_{S^1} = F$ (in the sense of part (f)), then show $u = v$. [This is a little tricky with no hint.]

On your next problem set, we will use problem 3 above to construct the inverse of the Laplacian on the unit disk. We will use our explicit construction to show that the inverse is a compact Hermitian operator, and then apply the spectral theorem to show that the eigenvalues of the Laplacian on the disk are a discrete set of negative reals, which diverge to infinity. This (with some physics) will explain why if you build a circular drum, frequencies it can produce are discrete, with a (strictly) positive lowest frequency the drum can produce. In this case it turns out that audible frequencies are multiples of this lowest frequency.