Assignment 7: Compact operators

Assigned Wed 05/16. Due Wed 05/23.

1. An operator $T \in B(X, X)$ is called *Hilbert-Schmidt* if there exists an orthonormal complete system $\{e_n\}$ of X, such that the series $\sum \|Te_n\|^2$ converges.
   (a) Show that Hilbert-Schmidt operators are compact. [HINT: Let $T_N(x) = \sum_1^N \langle x, e_n\rangle T(e_n)$. Show that T_N is compact, and $\|T - T_N\|_{B(X,X)}$ converges to 0.]
   (b) Find an example of an operator which is compact but not Hilbert-Schmidt.

2. Let $\triangle^{-1} : L^2([-\pi, \pi]) \to L^2([-\pi, \pi])$ be defined by $\triangle^{-1} f = -\sum_{n \neq 0} \frac{\hat{f}(n)}{n^2} e_n$. [If you choose to assume your functions are periodic on $[0, 1]$ (instead of $[-\pi, \pi]$), and define $e_n(x) = e^{2\pi i n x}$ (instead of $e_n(x) = e^{inx}$), then you should define $\triangle^{-1} f = \frac{-1}{4\pi^2} \sum_{n \neq 0}(\cdots)$ instead.]
   (a) If $f \in C^2$, and $\int f = 0$, then show that $(\triangle^{-1})\triangle f = f$. What happens if $\int f \neq 0$? [By C^2 here we mean functions which are periodic and have continuous second derivative. For simplicity, we've assume that f is a function of one variable. In this case, the Laplacian $\triangle f$ is just the second derivative f''.]
   (b) If $f \in H^s$ with $s > \frac{1}{2}$ with $\int f = 0$, then show that $\triangle^{-1} f \in C^2$, thus $\triangle(\triangle^{-1} f)$ is well defined. Further show $\triangle(\triangle^{-1} f) = f$.
   (c) Show that $\triangle^{-1} : L^2 \to L^2$ is compact and Hermitian.
   (d) Compute $\sigma(\triangle^{-1})$, and show that $\sigma(\triangle^{-1})$ consists of only eigenvalues. What are the eigenfunctions of $\triangle^{-1}$?
   (e) If you replace $\triangle$ with some elliptic differential operator L (see homework 5 question 1), show that you can define L^{-1} in a similar manner to make $L^{-1} : L^2 \to L^2$ compact (but not necessarily hermitian), such that L^{-1} is the inverse of L on some nice subset of L^2. Compute $\sigma(L^{-1})$.

3. (a) Let $1 \leqslant p < q \leqslant \infty$. Show that the inclusion map $i : \ell^p \to \ell^q$ is not compact. [Homework 2 4(d) shows that $\ell^p \subseteq \ell^q$, and that the inclusion is continuous.]
   (b) Let $1 \leqslant p < q \leqslant \infty$. Show that the inclusion map $i : L^q([0, 1]) \to L^p([0, 1])$ is not compact. [See homework 2 4(b) for the definition of this map. It also shows up in a few subsequent homeworks. One possible solution would be to let $\chi_n(x) = 1$ if the n^{th} digit in the binary expansion of x is 1, and let $\chi_n(x) = -1$ otherwise. Then $\|\chi_n\|_{L^p} = \|\chi_n\|_{L^q} = 1$ for all n, but $\{\chi_n\}$ is discrete in L^p. The χ_n's are called the Radamacher functions, and like the trigonometric basis e_n, the Radamacher functions form a orthogonal complete set in L^2. Simpler solutions to this problem exist.]

4. Let $A \in B(X, X)$. We define $e^A = \sum_0^\infty \frac{A^n}{n!}$.
   (a) Show that the series $\sum \frac{A^n}{n!}$ converges in $B(X, X)$.
   (b) If $A, B \in B(X, X)$ are such that $AB = BA$, show that $e^A e^B = e^{A+B}$.
   (c) We say $U \in B(X, X)$ is unitary if $U^* = U^{-1}$. If A is Hermitian, show that e^{iA} is unitary.

5. Let $K : [0, 1] \times [0, 1] \to \mathbb{C}$ be continuous and periodic, and define $T : L^p \to L^p$ by $Tf(x) = \int K(x, y) f(y) dy$ (see homework 6, 2(a)).
   (a) When $p = 2$, show that T is Hilbert-Schmidt (hence compact). [HINT: Let e_n be the Fourier basis of L^2. Show that $\sum \|Te_n\|_{L^2}^2 = \int |K(x, y)|^2 \, dx \, dy$.]

   T is still compact (though not Hilbert-Schmidt) when $p \neq 2$. We prove that here.
   (b) Let (K_n) be a sequence of continuous functions such that $(K_n) \to K$ uniformly (on $[0, 1]^2$). Let $T_n \in (L^p, L^p)$ be defined by $T_n f(x) = \int K_n(x, y) f(y) \, dy$. Show that $(T_n) \to T$ in $B(L^p, L^p)$.
   (c) Suppose $K_N \in \operatorname{span}\{e_n \mid n \in \mathbb{Z}^2, |n| \leqslant N\}$ (recall $e_n = e^{2\pi i \langle n, x\rangle}$). Show that the image of T_N is finite dimensional. [Here T_N is defined as in the previous subpart.]
   (d) Show that $T : L^p \to L^p$ is compact.

6. If $A, B \in B(X, X)$, show that $\sigma(AB) - \{0\} = \sigma(BA) - \{0\}$. [HINT: Show that $I - AB$ is invertible if and only if $I - BA$ is invertible. Use some clever geometric series trick to do this.]