Assignment 6: Dual spaces

Assigned Wed 05/09. Due Wed 05/16.

1. Show that Lemma 4 from the handout (solution of 3(b) from the midterm) is false if $\alpha = \beta$.
2. Let X be the completion of periodic C^1 functions with inner product defined by $\langle f, g\rangle = \int f\bar{g} + \int f'\bar{g}'$.
   (a) Show that there exists $D \in B(X, L^2)$ such that for all C^1 functions $f \in X$, $Df = f'$. Compute the adjoint of the operator iD.
   (b) Show that there does not exist $D \in B(L^2, L^2)$ such that for all C^1 functions $Df = f'$.
   (c) (Unrelated) Show that X is isomorphic to H^1.
3. (a) Let $K : [0,1] \times [0,1] \to \mathbb{C}$ be continuous. If $f \in L^p([0,1])$, then define Tf by $Tf(x) = \int K(x,y)f(y)\,dy$. Show that $T \in B(L^p, L^p)$. When $p = 2$, compute T^*.
   (b) Let $f \in L^1([0,1])$. Define $T : L^p \to L^p$ by $T(g) = f * g$. When $p = 2$, compute T^*.
4. (a) Show that $(\ell^1)^*$ is (isometrically) isomorphic to ℓ^∞.
   (b) Let $c_0 \subseteq \ell^\infty = \{(x_n)_n \mid (x_n)_n \to 0\}$. Show that $c_0^* \approx \ell^1$.
   (c) Suppose there exists $\Lambda \in (\ell^\infty)^*$ such that $\Lambda((x_n)_n) = \lim x_n$ whenever the limit exists. Show that there does not exist $(a_n) \in \ell^1$ such that $\Lambda(x) = \sum a_n x_n$ for all $x = (x_n)_n \in \ell^\infty$. [Some junk set theory will show the existence of such Λ. Hence this problem shows that $(\ell^\infty)^* \supsetneq \ell^1$.]
   (d) Let $1 < p < \infty$, and $q = \frac{p}{p-1}$. Let $\phi_p : \ell^q \to (\ell^p)^*$ be the isomorphism defined by $\phi(a)(x) = \sum a_i x_i$, where $a = (a_n) \in \ell^q$, and $x = (x_n) \in \ell^p$. For any Banach space X, let $i : X \to X^{**}$ be the canonical embedding defined by $i(x)(x^*) = x^*(x)$ for all $x^* \in X^*$. Show that $\phi_q^{-1} \circ \phi_p^*$ is the inverse of the canonical embedding $i : \ell^p \to (\ell^p)^{**}$. [Note that the arguments from class only say that $(\ell^p)^{**}$ is isomorphic to ℓ^p. There exist pathological examples of Banach spaces such that X^{**} is isomorphic to X, but the canonical embedding is not an isomorphism. When the canonical embedding is an isomorphism (which is generally a desirable thing), the Banach space is called *reflexive*.]
5. Define $\varphi : L^q \to (L^p)^*$ by $\varphi(f)(g) = \int fg$. Show that φ is an isometric embedding. [φ is actually an isomorphism, but I don't know a proof of this without using the Radon-Nikodym theorem.]
6. Using the axiom of choice one can show that for any Banach space X, $\|x\|_X = \sup_{\|x^*\|_{X^*}=1} |x^*(x)|$. (Note the 'duality' of this statement with the definition of the norm in X^*: $\|x^*\|_{X^*} = \sup_{\|x\|_X=1} |x^*(x)|$). Since I can't in good conscience prove this in class, you prove this explicitly for the following Banach spaces:
   (a) ℓ^p, when $1 \leqslant p \leqslant \infty$.
   (b) L^p, when $1 \leqslant p \leqslant \infty$.
7. *(Youngs inequality)* Suppose $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f, g \in C^0$, show that $\|f * g\|_{L^r} \leqslant \|f\|_{L^p}\|g\|_{L^q}$. [Hint: This is quite hard to prove directly. The trick here is to pick any $h \in C^0([0,1])$, and show that $|\int (f*g)(x)h(x)\,dx| \leqslant \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^{r'}}$, where $r' = \frac{r}{r-1}$, and then use the previous problem.]