

Homework 5: Discrete Sobolev spaces

Assigned Fri 05/04. Due Wed 05/09

Since you already had a (possibly) traumatic midterm to deal with this week, this homework is short.

1. Let $L : C^2 \rightarrow C^0$ be defined by

$$Lu = \sum_{i,j=1}^d a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} + cu$$

where a_{ij} , b_i and c are (given) constants, and $a_{ij} = a_{ji}$. The operator L is called *elliptic* if there exists $\lambda > 0$ such that for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, we have $|\sum_{i,j} a_{ij} \xi_i \xi_j| \geq \lambda |\xi|^2$. If L is elliptic, $Lu = g$ and $g \in H^s$, show that $u \in H^{s+2}$. Conclude that if $g \in C^\infty$, then $u \in C^\infty$. [Note that $Lu = \Delta u$ is elliptic, and hence what we did in class is a special case of the above. This is usually known as elliptic regularity.]

2. Pick $0 \leq s < t$. Let $S = \{f \in H^t \mid \|f\|_{H^t} \leq 1\}$. Show that $\bar{S} \subseteq H^s$ is compact. [This is a usual feature in functional analysis. When you are working with a space of functions, the method by which you obtain compact sets is to impose some restriction on the derivatives! This is something you saw on your midterm too with C^α . The proof here is not as horrific as the midterm question. Use something similar to the last problem on the first assignment. . . .]
- ★ 3. Let $s > \frac{d}{2}$. Suppose $\alpha = s - \frac{d}{2} \in (0, 1)$, and $f \in H^s([0, 1]^d)$, then show that $f \in C^\alpha([0, 1]^d)$.
- ★ 4. (*Poincaré Inequality*) Say $f \in C^1[-\pi, \pi]$ and $\int_{-\pi}^{\pi} f = 0$. Show that $\|f\|_{L^2} \leq \|f'\|_{L^2}$. [If this is too easy, try proving it without Fourier Series! In this case you will only be able to show $\exists c \ni \|f\|_{L^2} \leq c \|f'\|_{L^2}$.]

Midterm

Assigned Thu 05/03 2:15PM. Due Fri 05/04 2:15PM.

If you have trouble with 3(b), or 4(b) don't lose sleep over it. Come back to them if you have time: 3(b) is for the most part straightforward (if you follow the hint), but has some pretty horrific details. 4(b) is short and elegant, however if you don't see the proof I have in mind, it is likely you will spew some 300 pages and get no where near the correct solution.

1. Let X be an inner product space. Suppose $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal set such that for any $x \in X$, $\|x\|^2 = \sum |\langle x, e_n \rangle|^2$. Show that for any $x, y \in X$, $\langle x, y \rangle = \sum \langle x, e_n \rangle \overline{\langle y, e_n \rangle}$.
2. Let $X = \{f \mid f : [0, 1] \rightarrow \mathbb{C} \text{ is continuous}\}$. For $f, g \in X$, define $\langle f, g \rangle = \int_0^1 f \bar{g}$. I've mentioned before that X is an inner product space, but not Hilbert (in fact we defined L^2 to be the completion of X). Give an explicit example of a Cauchy sequence in X which is not convergent.
3. Let $\alpha \in (0, 1)$. Let $C^\alpha([0, 1])$ be the set of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ such that $\sup\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} \mid x, y \in [0, 1] \text{ and } x \neq y\} < \infty$.
 - (a) If $f \in C^\alpha$, we define $\|f\|_{C^\alpha} = \|f\|_{L^\infty} + \sup\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} \mid x, y \in [0, 1] \text{ and } x \neq y\}$. Show that C^α with norm $\|f\|_{C^\alpha}$ is a Banach space. [In interest of saving trees, only check that the triangle inequality is satisfied, and that Cauchy sequences converge.]
 - (b) Pick $\beta > \alpha$, and let $S = \{f \in C^\alpha \mid \|f\|_{C^\beta} \leq 1\}$. Show that S is a compact subset of C^α . [NOTE: You need $\beta > \alpha$. S is not a compact subset of C^β .]
HINT: Let C_n be the set of all piecewise linear functions such that $\forall k \in \{0, \dots, n\}$, $f(\frac{k}{n}) \in \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$. Note C_n is finite. Show that for any $\varepsilon > 0$, there exists n such that $S \subseteq \cup_{f \in C_n} B_\varepsilon(f)$. This implies total boundedness of S .
4. Let (ϕ_n) be an approximate identity.
 - (a) For any $k \in \mathbb{Z}$, show that $\lim_{n \rightarrow \infty} \hat{\phi}_n(k)$ exists.
 - (b) Let $F : L^1 \rightarrow \ell^\infty$ be defined by $F(f) = (\hat{f}(n))_n$ (as in Homework 4 problem 6). Show that $(F(\phi_n))_n$ does not converge in ℓ^∞ .