Homework Assignment 3: Hilbert and Banach spaces

Assigned Wed 04/18. Due Wed 04/25.

1. Let X be the set of (complex valued) sequences which have only *finitely many* non-zero terms. If $x = (x_n)_n$, $y = (y_n)_n \in X$ define $\langle x, y \rangle = \sum x_k \overline{y_k}$. Note that this sum has only finitely many non-zero terms, and hence is always finite. It is easy to see that this make X an inner product space (but not a Hilbert space).
   (a) Let $V = \{(x_n)_n \in X \mid \sum_k \frac{x_k}{k} = 0\}$. Show that V is a closed subspace of X, but $X \neq V \oplus V^\perp$. [Hint: Show $V \subsetneq X$, but $V^\perp = \{0\}$.]
   (b) Let $V = \{(x_n)_n \in X \mid \sum_k \frac{x_k}{\sqrt{k}} = 0$. Show that $\bar{V} = X$. [Note the similarity of the definitions of V in this and the previous subpart. Yet the subspace in the previous subpart is closed, and the one in this subpart is dense.]

2. (a) Show that the closure of a convex set is convex.
   (b) Let X be a Banach space, and $S \subseteq X$. We define the *convex hull* of S (denoted by $\operatorname{co}(S)$) to be $\{\sum_1^n c_i x_i \mid x_i \in S, c_i \in [0,1], \sum c_i = 1\}$. We define *convex closure* of S, denoted by $\overline{\operatorname{co}}(S)$, to be the closure of the convex hull of S. Show that $\operatorname{co}(S)$ and $\overline{\operatorname{co}}(S)$ are convex.

3. Let $X = \{f : [0,1] \to \mathbb{C} \mid f \text{ is continuous.}\}$, with $\|f\| = \sup\{|f(x)| \mid x \in [0,1]\}$.
   (a) Find a closed convex subset of X which has more than one element of smallest norm.
   (b) Find a closed convex subset of X which does *not* have an element of smallest norm. [Hint: If cooking one up off the top of your head yields only the unhealthy habit of eating worms, here's a suggestion: Let $S = \{f \in X \mid f(0) = 0, \int_0^1 f = 1\}$.]

4. This problem requires some prior knowledge of countability. If you don't know it, it's not the end of the world if you skip this problem. Recall a Hilbert (or Banach) space X is called *seperable* if there exists a countable dense subset.
   (a) Show that ℓ^p is a seperable. [Hint: Let $S = \{\sum_1^N (p_j + iq_j)e_j \mid N \in \mathbb{N}, p_j, q_j \in \mathbb{Q}\}$. Showing S is dense in X is easy enough. If you don't know that S is countable, you should convince yourself (or ask me).]
   (b) Define ℓ^∞ to be the set of all *bounded* sequences. If $x = (x_n)_n \in \ell^\infty$, then define $\|x\|_{\ell^\infty} = \sup\{|x_n| \mid n \in \mathbb{N}\}$. Show that ℓ^∞ is *not* seperable. [Hint: Find an uncountable set $S \subseteq \ell^\infty$ such that $s, t \in S \implies \|s - t\|_{\ell^\infty} = 1$ when $s \neq t$.]
   (c) Show that $L^p([0,1])$ is seperable. [Hint: This is a little cumbersome but straightforward. Let S be the set of all piecewise linear functions such that $f(q) \in \mathbb{Q}$ whenever $q \in [0,1] \cap \mathbb{Q}$. Use uniform continuity to show that for any continuous function f and $\varepsilon > 0$ there exists an element $s \in S$ such that $\|s - f\|_{L^\infty} < \varepsilon$. This should immediately imply S is dense in $L^p([0,1])$.]

5. Give an example of two Banach (or Hilbert) spaces X, Y and a continuous linear function $T : X \to Y$ such that the image of T is not a closed subspace of Y. [Hint: An example of this can be found on last weeks homework. If you look closely enough, question 1(b) from this homework can also be suitably modified to construct such an example (this is more work than quoting from last weeks homework)]

6. *(Quotient spaces)* Let X be a Banach space, and $V \subseteq X$ a closed subspace. Recall $X/V = \{x + V \mid x \in X\}$, where $x + V = \{x + v \mid v \in V\}$ are the cosets. We define $\|x + V\| = \inf\{\|x + v\| \mid v \in V\}$. [If you haven't seen quotient vector spaces in 113, you might find the above description too brief. Look it up in any Linear Algebra book (e.g. the ones listed online)]
   (a) Show that this norm makes X/V a normed space, and the quotient map $\varphi : X \to X/V$ defined by $\varphi(x) = x + V$ is continuous.
   (b) If V is not closed, then show that X/V is not even a normed space. Further show that the quotient map is not continuous.
   (c) If X is a Hilbert space, how would you define an inner product on X/V to make it a Hilbert space, so that the quotient map continuous? [Note: For $x, y \in X$, the set $\{\langle x + v, y + v' \rangle \mid v, v' \in V\}$ is not bounded. So don't write a proof that involves the inf or sup of a similar set]

(d) Suppose X/V is finite dimensional. Then show that there exists a closed subspace $U \subseteq X$ such that $U \oplus V = X$.

Let X be a Banach space and $V \subseteq X$ a closed subspace. We say $U \subseteq X$ is a compliment of V if U is a *closed* subspace and $X = U \oplus V$. The previous subpart says that any subspace of finite co-dimension (i.e. X/V has finite dimension) has a compliment. If X is a Hilbert space, we know $X = V \oplus V^\perp$, so any closed subspace of a Hilbert space has a compliment. However but this is not true for Banach spaces. There's a 'deep' theorem (by two people I can't remember the names of) saying that if every closed subspace of a Banach space has a compliment, then X is isomorphic to a Hilbert space.

The next two subparts are (considerably) harder. If you can't figure it out just remember that such examples exist, and in general Banach spaces you shouldn't take for granted what might otherwise seem obvious.

(e) Find an example of a Banach space X and a closed subspace V which does not have a compliment.

(f) Show that X/V is a Banach space. [The original version of this question was wrong! The proof is not that hard, and is online.]

1. Proof of the Riesz-Frechet Theorem

Since I was brief in class, here's the expanded version

Theorem 1 (Riesz-Frechet)**.** *Let X be a Hilbert space, and $\Lambda : X \to \mathbb{C}$ a bounded linear functional. Then $\exists v \in X$ such that $\forall x \in X$, $\Lambda(x) = \langle x, v \rangle$.*

Proof. Let $U = \ker(\Lambda)$. Since Λ is bounded (hence continuous), $U = \Lambda^{-1}(0)$ is closed. Thus $X = U \oplus U^\perp$.

If Λ is the 0 functional, we choose $v = 0$ and finish the proof. If not, notice (from 113) that X/U is isomorphic to the image of Λ, which must be $\mathbb{C}$. Further since $X = U \oplus U^\perp$, $X/U \cong U^\perp$, and hence $U^\perp \cong \mathbb{C}$ is one dimensional.

Pick any $w \in U^\perp$ and let $v' = \frac{w}{\Lambda w}$. Note $\Lambda v' = 1$. We claim our desired vector is given by $v = \frac{v'}{\|v'\|^2}$. We see this as follows: Let $x \in X$. Since $X = U \oplus U^\perp$, and $U^\perp$ is one dimensional, there exists a unique $u \in U$ and $\alpha \in \mathbb{C}$ such that $x = u + \alpha v'$. Thus

$$\Lambda(x) = \Lambda(u) + \alpha \Lambda(v') = \alpha$$

and

$$\langle x, v \rangle = \langle u + \alpha v', v \rangle = 0 + \alpha \left\langle v', \frac{v'}{\|v'\|^2} \right\rangle = \alpha$$

finishing the proof. ∎

Remark 2. If this business of quotienting above did not make sense, here's a shorter (less transparent) proof that does not rely on quotients.

Assume Λ is not the 0 functional, and let $U = \ker(\Lambda)$ be as above. Since $X = U \oplus U^\perp$, we know $U^\perp$ is not $\{0\}$, so pick any $v' \in U^\perp$ with $\|v'\| = 1$. We claim $v = \overline{\Lambda v'} v'$ is our desired vector.

To see this, pick any $x \in X$, and let $u = (\Lambda x)v' - (\Lambda v')x$. Note $\Lambda u = (\Lambda x)(\Lambda v') - (\Lambda v')(\Lambda x) = 0$, and thus $u \in U$. Hence $\langle u, v' \rangle = 0$. This gives

$$\begin{aligned} & (\Lambda x) \langle v', v' \rangle = (\Lambda v') \langle x, v' \rangle \\ \implies & \qquad \Lambda x = \left\langle x, \overline{\Lambda v'} v' \right\rangle \\ & \qquad\quad = \langle x, v \rangle \end{aligned}$$

Remark 3. If not apparent, the crucial ingredient in the proofs above is that if $U \subseteq X$ is a closed subspace which is not all of X, then $U^\perp$ is not the 0 subspace. This is of course false if we only dealt with inner product spaces, and not Hilbert spaces (Question 1).