# Homework Assignment 2: Cauchy completion and $L^p$

Assigned Wed 04/11. Due Wed 04/18.

1. Let $X$ be a normed space, and $\bar{X}$ it's completion as defined in class (the set of all equivalence classes of Cauchy sequences). Prove the following.
   (a) If $\bar{x}, \bar{y} \in \bar{X}$ then either $\bar{x} \cap \bar{y} = \emptyset$ or $\bar{x} = \bar{y}$.
   (b) Let $(x_n)$ be a Cauchy sequence in $X$, and $(x_{n_k})$ any subsequence. Show that $\overline{(x_{n_k})} = \overline{(x_n)}$. [It is enough to show that $(x_{n_k}) \in \overline{(x_n)}$.]

   Suppose additionally $X$ is an inner product space. If $\bar{x}, \bar{y} \in \bar{X}$, we pick Cauchy sequences $(x_n) \in \bar{x}$ and $(y_n) \in \bar{y}$. Define $\langle \bar{x}, \bar{y} \rangle = \lim \langle x_n, y_n \rangle$.
   (c) Show that $\langle \cdot, \cdot \rangle$ is defined. Namely if $(x_n), (y_n)$ are Cauchy sequences in $X$, then $\lim \langle x_n, y_n \rangle$ exists.
   (d) Show that $\langle \cdot, \cdot \rangle$ is *well* defined. Namely if $(x_n), (x'_n) \in \bar{x}$ and $(y_n), (y'_n) \in \bar{y}$, then $\lim \langle x_n, y_n \rangle = \lim \langle x'_n, y'_n \rangle$.
   (e) Show that $\langle \cdot, \cdot \rangle$ defines an inner product on $\bar{X}$.

   Recall we defined (or will define) $\sigma : X \to \bar{X}$ by $\sigma(x) = \overline{(x)}$ (i.e. $\sigma(x)$ is the equivalence class of the constant sequence $(x)$). A crucial part of the theorem of completions from class is that $\sigma : X \to \bar{X}$ is a linear isometry, and the image $\sigma(X)$ is dense in $\bar{X}$. We're never going to prove this in class, so these statements sneakily find their way into your homework.
   (f) Show that $\sigma$ is a linear isometry. [Recall $\sigma$ is an isometry if $\forall x \in X$, $\|\sigma(x)\| = \|x\|$.]
   (g) If $X$ is an inner product space, show additionally that $\sigma$ is inner product preserving: i.e. $\forall x, y \in X$, $\langle \sigma(x), \sigma(y) \rangle = \langle x, y \rangle$. [NOTE: It's easy to show that any linear isometry has this property]
   (h) Let $(x_n)$ be a Cauchy sequence in $X$. Show that $(\sigma(x_n))$ is a Cauchy sequence in $\bar{X}$, and converges to $\overline{(x_n)}$. Conclude $\sigma(X)$ is dense in $\bar{X}$. [Recall $A$ is called a dense subset of $X$ if the closure of $A$ is $X$.]

2. Let $X$ be a normed space. Suppose $\bar{X}$ is *any* Banach space such that there exists $\sigma : X \to \bar{X}$ which is linear, isometric (i.e. $\forall x \in X$, $\|\sigma(x)\| = \|x\|$) and has dense image (i.e. $\sigma(X)$ is dense in $\bar{X}$).
   (a) Let $Y$ be a Banach space, and $f : X \to Y$ be a uniformly continuous function on $X$. Show that there exists a unique function $\bar{f} : \bar{X} \to Y$ such that $\bar{f} \circ \sigma = f$. [If $X$ as a subset of $\bar{X}$, and $\sigma$ is the inclusion map, then this tells you that any uniformly continuous function defined on $X$ extends uniquely to a uniformly continuous function defined on $\bar{X}$. It might help you internalize this if you first thought of an example where $X = \mathbb{Q}$, and $\bar{X} = \mathbb{R}$. The proof of this has nothing to do with the explicit construction of $\bar{X}$ we have. The hint is to prove and use the fact that uniformly continuous functions take Cauchy sequences to Cauchy sequences]
   (b) *(Uniqueness of completions)* Suppose $\bar{X}'$ is a Banach space such that there exists $\sigma' : X \to \bar{X}'$ which is linear, isometric and with dense image. Show that $\bar{X}$ and $\bar{X}'$ are isometrically isomorphic (i.e. there exists $\phi : \bar{X} \to \bar{X}'$ which is linear, invertible, and isometric). [This explains the 'uniqueness' part in the statement of the theorem about completions: Any two spaces with the properties above are isomorphic, by this problem. Hence we say that the completion is *'unique up to isomorphism'*.]
   (c) If $f$ is only continuous, but not uniformly continuous, the above is false. Here's an example. Let $X = \mathbb{Q}$, $\bar{X} = \mathbb{R}$, and $\sigma : \mathbb{Q} \to \mathbb{R}$ be the inclusion map. We define $f : \mathbb{Q} \to \mathbb{R}$ by $f(x) = \frac{1}{x^2 - 2}$. Show that $f$ is continuous (duh) but does not extend to a continuous function on $\mathbb{R}$.

3. Let $X, Y$ be normed spaces.
   (a) Show that any linear continuous function $T : X \to Y$ is uniformly continuous.
   (b) Let $T : X \to Y$ be linear. Show that $T$ is continuous if and only if $\exists c > 0$ such that $\|T(x)\| \leqslant c\|x\|$.
   (c) Finally here's an example of a *linear* function that is not continuous (contrast this with the finite dimensional case, where all linear functions are given by multiplication by some matrix, which is as continuous as you can get): Let $X$ be the set of continuous functions on $[0, 1]$, endowed with the $L^p$ norm. If $f \in X$, define $T(f) = f(0)$. Show that $T : X \to \mathbb{C}$ is not continuous.

4. Recall $L^p([0, 1])$ is the completion of the set of all continuous functions on $[0, 1]$ with respect to the norm $\|f\|_{L^p} = (\int_0^1 |f|^p)^{1/p}$. Suppose now $1 \leqslant p < q$.

(a) Let $f : [0,1] \to \mathbb{C}$ be continuous. Show that $\|f\|_{L^p} \leqslant \|f\|_{L^q}$. [HINT: Use Jensen's inequality]

(b) Conclude that there exists a continuous linear function $T : L^q[0,1] \to L^p[0,1]$ such that for every continuous function $f$, $T(f) = f$. [Recall $L^p$ is completion of the set of continuous functions with respect to the $L^p$ norm. This problem tells you that if $p < q$, $L^q$ can be thought of as a subset of $L^p$.]

If the domain of your function does not have finite length, then the above is false

(c) Let $C_c(\mathbb{R})$ be the set of continuous compactly supported functions (i.e. functions which are 0 outside some interval $[a,b]$). We define $L^p(\mathbb{R})$ to be the completion of $C_c(\mathbb{R})$ with respect to the norm $\|f\|_{L^p} = (\int_{-\infty}^{\infty} |f|^p)^{1/p}$. Find a sequence $(f_n) \in C_c(\mathbb{R})$ such that for all $n \in \mathbb{N}$, $\|f_n\|_{L^q} = 1$, but $\lim_{n\to\infty} \|f_n\|_{L^p} = \infty$. Conclude there is no *continuous* linear function $T : L^q \to L^p$ such that $T(f) = f$ for all continuous compactly supported functions $f$. [HINT: The intuition here is that when $x$ is small, $x^q \ll x^p$. Thus you should be able to find some power $s$ such that $\int_1^\infty (\frac{1}{x^s})^p = \infty$, but $\int_1^\infty (\frac{1}{x^s})^q < \infty$. Now let $f_n$ be a continuous function which is 0 outside $[1,n]$, and is $\frac{1}{x^s}$ on $[1+\frac{1}{n}, n-\frac{1}{n}]$, scaled appropriately to make the $L^q$ norm 1. ]

Finally, we mention that the *reverse* happens with $\ell^p$.

(d) If $x \in \ell^p$, show that $\|x\|_{\ell^q} \leqslant \|x\|_{\ell^p}$. Conclude that $\ell^p \subseteq \ell^q$, and the inclusion map is continuous.

5. Let $H \subseteq \ell^2$ be the set of all sequences $(x_n)$ such that $\sum n^2 |x_n|^2 \leqslant 1$. Show that $H$ is a compact subset of $\ell^2$. [If you can read this you don't need glasses. You probably also realize that you're not getting a hint on this one.]