Math 135 Spring 2007 :: Final exam :: Tuesday June 12

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

Print your name and student ID number, and write your signature to indicate that you accept the honor code. The exam starts at 8:30a and ends at 11:30a. Books and notes are allowed, but not calculators, phones, or computers. There are 6 problems for a total of 100 points.

Question	Score	Maximum
1		15
2		10
3		30
4		25
5		10
6		10
Total		100

1. Multiple choice questions. Each correct answer: 3 pts. Each wrong answer: 0 pt. No need to justify your answers. In what follows, "smooth" means infinitely differentiable.

   (a) Can a conservative system have a smooth Liapunov function?

   Yes, they all do   Yes, but not all do   No

   (b) Can a reversible system have a smooth Liapunov function?

   Yes, they all do   Yes, but not all do   No

   (c) Consider a map $x_{n+1} = f(x_n)$, with f smooth. Assume that x^* is a fixed point, and $f'(x^*) = -2$. Is x^* attracting?

   Yes   Cannot conclude   No

   (d) Consider a flow $\dot{x} = f(x)$, with f smooth. Assume that x^* is a fixed point, and $f'(x^*) = -2$. Is x^* attacting?

   Yes   Cannot conclude   No

   (e) Consider a smooth 2D flow defined in the whole plane. Consider a closed, bounded region D that is trapping for the flow, and that does not contain any fixed point. Does D necessarily contain an attracting limit cycle?

   Yes   No

2. Consider the system

$$\begin{cases} \dot{x} = f(y), \\ \dot{y} = g(x), \end{cases}$$

where f and g are two arbitrary, continuous functions.

   (a) (5 pts) Exhibit a conserved quantity for this system.

   (b) (5 pts) Show that the evolution preserves areas in phase-space.

3. One of the most important applications of the concepts seen in class is to the modeling of neuron firing, where one tries to explain how a neuron can generate electric pulses. The FitzHugh-Nagumo equations are a realistic two-dimensional model for that:

$$\begin{cases} \dot{u} = u - \frac{u^3}{3} - v + I, \\ \dot{v} = \epsilon(2 + 2u - v). \end{cases}$$

Here u is the membrane voltage, I is the current excitation, and v is related to the opening of Na^+ and K^+ ion channels. In this exercise, we will establish the existence of oscillatory solutions when the excitation I is large enough, but not too large.

In what follows we will fix $\epsilon = \frac{5}{9}$.

(a) (5 pts) Find and classify all the fixed point(s) in the case $I = 2$.

(b) (5 pts) Plot the nullclines and deduce qualitative information about the phase portrait, in the case $I = 2$.

(c) (10 pts) Prove the existence of a closed orbit in the case $I = 2$.

(d) (5 pts) *Without finding the fixed point(s) explicitly*, rule out the possibility of a bifurcation of fixed points as I varies.

(e) (5 pts) Bifurcations nevertheless occur as I varies. Discuss their type, and find the values of I at which they occur.

4. Consider the map $x_{n+1} = f(x_n)$ where $f(x) = x^2 + rx$. As usual, r is a real parameter.

(a) (5 pts) Find the fixed point(s). For which values of r do they exist?

(b) (5 pts) Determine the stability of the fixed point(s) as a function of r.

(c) (5 pts) Find the 2-cycle(s), as a function of r. For which values of r do they exist?

(d) (5 pts) Determine the stability of the 2-cycle(s) as a function of r.

(e) (5 pts) Plot a partial bifurcation diagram based on the information obtained. Discuss the type of bifurcations that occur.

5. (10 pts) Consider the system

$$\begin{cases} \dot{\theta}_1 = 1, \\ \dot{\theta}_2 = \pi, \end{cases}$$

with θ_1 and θ_2 in $[0, 2\pi)$.

Why isn't this system considered chaotic?

6. (a) (5 pts) Find the similarity dimension of the Cartesian product of the middle-thirds Cantor set with itself. (The Cartesian product $A \times B$ of two sets A and B is defined as the set of couples (a, b) such that $a \in A$ and $b \in B$.)

   (b) (5 pts) Find the similarity dimension of the subset of $[0, 1]$ consisting of real numbers with only odd digits in their decimal expansion.