A short note on the first problem from last quarter's homework:

Problem: Find all radially symmetric harmonic functions, in $\mathbb{R}^n$

Sol: We want to solve $\Delta u = 0$ in $\mathbb{R}^n$

If u is radially symmetric, then we know that

$$u(x) = f(|x|) \quad (\text{here } x \in \mathbb{R}^n \ \& \ |x| = \text{length of } x)$$

Say $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Then $\Delta u = \sum_1^n \frac{\partial^2}{\partial x_i^2} f(|x|)$

Note that $\partial_{x_i} f(|x|) = f'(|x|) \cdot \frac{\partial}{\partial x_i} \sqrt{x_1^2 + \dots + x_n^2}$

$$= f'(|x|) \frac{x_i}{|x|}$$

$$\Rightarrow \frac{\partial^2}{\partial x_i^2} f(|x|) = f''(|x|) \frac{x_i^2}{|x|^2} + f'(|x|) \left(\frac{|x| - x_i \left(\frac{x_i}{|x|}\right)}{|x|^2} \right)$$

$$\Rightarrow \Delta u = \sum_1^n \frac{\partial^2}{\partial x_i^2} f(|x|) = f''(|x|) + \frac{f'(|x|)}{|x|^2} \left(n|x| - |x| \right)$$

$$= f''(|x|) + \frac{(n-1)}{|x|} f'(|x|) = 0$$

Put $r = |x|$ & $g(r) = f'(r) \Rightarrow g'(r) + \frac{(n-1)}{r} g(r) = 0$

$$\Rightarrow \ln g' = -(n-1) \ln r + \ln c_1 \Rightarrow g(r) = \frac{c_1}{r^{n-1}}$$

Since $f' = g$, integrating gives

$$u(x) = f(|x|) = \begin{cases} c_1 \ln|x| + c_2 & \text{when } n = 2 \\ \frac{c_1}{|x|^{n-2}} + c_2 & \text{when } n > 2 \end{cases}$$