Assignment 4: Assigned Thu 01/31. Due Thu 02/07

The strong maximum principle when you have non-constant boundary conditions works a little differently that you would expect. The version Strauss states, and the version I stated in class only work if your boundary conditions are constant in time (a point which I neglected to emphasize, and which Strauss *ignores* completely). I rectify this below. (Note however that the weak maximum principle of course does not have any such restriction on boundary conditions).

1. Let $h(t) = 0$ if $t \in [0,1]$, and $h(t) = 1-t$ if $t > 1$. Suppose u satisfies the PDE $\partial_t u - \nu \partial_{xx}^2 u = 0$ when $x \in (0,1)$, $t > 0$, with boundary conditions $u(0,t) = u(1,t) = h(t)$ and initial data $u(x,0) = 0$.
   (a) What is the maximum value of u on the rectangle $x \in [0,1]$, $t \in [0,2]$?
   (b) At which points does u attain this maximum value? Is u identically constant?
   (c) In light of the above example, how would you state the strong maximum principle if your boundary conditions are not constant in time?
2. Prove the following generalized forms of the weak maximum principle.
   (a) Let $T > 0$, $x_1, x_2 \in \mathbb{R}$, and a, b be two functions such that $a(x,t) \geqslant 0$ (with no assumption on b). Suppose u is a function such that $\partial_t u + b\partial_x u - a\partial_{xx}^2 u \leqslant 0$ on the rectangle $(x_1, x_2) \times (0, T)$, and u is continuous on the rectangle $[x_1, x_2] \times [0, T]$. Show that u attains it's maximum on the sides or bottom of this rectangle.
   (b) If instead $\partial_t u + b\partial_x u - a\partial_{xx}^2 u \geqslant 0$, show that u attains it's minimum on the sides or bottom of this rectangle.
3. Check that the heat equation has an infinite speed of propagation, in the following sense: If for any $\delta > 0$, we define $f(x) = 1$ when $|x| < \delta$ and $f(x) = 0$ otherwise. Let $u(x,t)$ solve $\partial_t u - \nu \partial_{xx}^2 u = 0$ for all $x \in \mathbb{R}$, $t > 0$ with initial data $u(x,0) = f(x)$. Then show that for any $t > 0$, $u(x,t) \neq 0$ for all x. [Thus the small heat source centered at 0 is *immediately* felt at all points x.]
4. **Section 2.4** 2, 16, 18, 19. [For 4, look up the error function in the text. Also note that Strauss uses $S(x,t)$ where I used $G(x, 2\nu t)$. Using G (for Gaussian) to denote this is more standard, and G is often called the heat kernel.]
5. In class I used the method of characteristics to derive that the general solution of $u_{tt} - c^2 u_{xx} = 0$ is of the form $u(x,t) = f(x - ct) + g(x + ct)$. Derive the same formula using the coordinate method. [Hint: Use coordinates given by $\eta = x - ct$ and $\xi = x + ct$.]
6. **Section 2.1** 11 [Hint: First solve it with 0 on the right hand side (factoring the operator like we did in class will help). Next guess a particular solution, and use the fact that the general solution of a linear inhomogeneous PDE is the general solution of the homogeneous part plus a particular solution (just like in ODE's).]

Assignment 5: Assigned Thu 02/07. Due Thu 02/14

Note you do not have to know the material in 3.3, 3.4 which references 3.1 and 3.2.

1. **Section 2.1** 5, 10
2. **Section 2.2** 1, 5, 6
3. **Section 3.4** 2, 5, 6
4. **Section 3.3** Covered, but no problems assigned.
5. (a) Compute the solution of $\partial_t u - \frac{1}{2}\partial_{xx}^2 u = f$ given $u(x,0) = 0$, and $f(x,t) = 1$ if $|x| \leqslant 1$, and $f(x,t) = 0$ otherwise.
   (b) Compute $\lim_{t \to \infty} \frac{1}{t^2} u(x,t)$.
   (c) *(Harder)* Compute compute $\lim_{t \to \infty} \frac{1}{t} u(x,t)$.
6. Suppose $D \subseteq \mathbb{R}^3$, and $\partial_{tt}^2 u - c^2 \triangle u = 0$ when $x \in D$, $t \in (0, \infty)$. Under what boundary conditions for u can you guarantee that $E(t) = \int_D ((\partial_t u)^2 + c^2 (\partial_x u)^2)$ is constant in time.