Math 131 Midterm

Tue, October 23, 2007

Time: 75 mins
Total: 50 points

This is a closed book test. Calculators can be kept handy for your 'moral comfort', but they will not be of any use at all. Please don't use cell phones or pagers. Good luck ☺

[10] 1. Solve the PDE $u_x + u_y = u^3$, given $u(0, y) = f(y)$. Is the solution (from the previous part) defined on the entire xy-plane? Justify.

Solution. First find the characteristics: $\frac{dy}{dx} = 1$, and hence $y - x = c$ are the characteristics, for any constant c.

Now put $v(t) = u(t, t + c)$. Then $\frac{dv}{dt} = u_x(t, t + c) + u_y(t, t + c) = u(t, t + c)^3 = v(t)^3$. Thus $dv = v^3\, dt$. Solving this ODE gives

$$\frac{-1}{2v^2} = t + c_2$$

for some constant c_2. Setting $t = 0$, we see that $c_2 = \frac{-1}{2u(0,c)} = \frac{-1}{2f(c)^2}$.

Thus finally if we put $x = t$, and $y = t + c$, we get

$$u(x, y) = \pm\sqrt{\frac{-1}{2(t + c_2)}} = \pm\sqrt{\frac{-1}{2x - \frac{1}{f(y-x)^2}}} = \frac{\pm f(y - x)}{\sqrt{1 - 2xf(y - x)^2}}$$

To find the region in the plane where the solution is defined, be careful: You would be tempted to say that this region is the region spanned by all the characteristics which touch the y axis. This is in general true only when your PDE is a *linear* PDE. Our PDE has u^3 on the right, and is certainly not linear. So perhaps a more careful inspection is required.

From our explicit solution, we see that $1 - 2xf(y - x)^2$ had better be positive for the solution to be defined. Thus the region in the plane where the solution u is defined is $\{(x, y) \mid 2|x|f(y - x)^2 \leqslant 1\}$. (Note that this certainly includes the y axis, and that if you follow the solution along any characteristic, it will certainly blow up for large x, provided f was non-zero. To get a better feel for this, take some specific example of f, and sketch this region.) □

[5] 2. (a) Suppose v satisfies the wave equation $v_{tt} - c^2 v_{xx} = 0$. If $v(x, 0) = v_t(x, 0) = 0$ whenever $|x| \leqslant 1$, then find the largest region in the xt-plane with $t \geqslant 0$, where you can guarantee that $v(x, t) = 0$.

Solution. Let $f(x) = v(x, 0)$, and $g(x) = v_t(x, 0)$. By D'Alembert, we know $v(x, t) = \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2}\int_{x-ct}^{x+ct} g$. Thus if both $x + ct$ and $x - ct$ are in the interval $[-1, 1]$, we are sure that v will be 0. So we can certainly guarantee that $v(x, t) = 0$ for all points such that $0 \leqslant ct \leqslant 1 - |x|$. □

Remark. Note that you should have *already known* this before you did out the algebra: We know that the wave equation has finite speed of propagation c. Thus all points who *only* feel the initial data in the interval $[-1, 1]$, are the only points we can guarantee to be 0. Since the initial data's information is transmitted with at most speed one, all such points must be exactly $0 \leqslant ct \leqslant 1 - |x|$. (You could also arrive at this by drawing the characteristics and staring at them.)

[5] (b) Let f be some function of x and t. Suppose u_1 and u_2 are two solutions of the forced wave equation $u_{tt} - c^2 u_{xx} = f$, with initial data $u_1(x, 0) = \varphi_1(x)$, $\partial_t u_1(x, 0) = \psi_1(x)$, $u_2(x, 0) = \varphi_2(x)$, $\partial_t u_2(x, 0) = \psi_2(x)$ respectively. Suppose whenever $|x| \leqslant 1$, $\varphi_1(x) = \varphi_2(x)$ and $\psi_1(x) = \psi_2(x)$. Find the largest region in the xt-plane (with $t \geqslant 0$) where you can guarantee $u_1(x, t) = u_2(x, t)$. [Hint: This is part (b) of a question.]

Solution. Step 0: Read the hint. Set $v = u_1 - u_2$. Then

$$v_{tt} - c^2 v_{xx} = (\partial_t t - c^2 \partial_{xx})(u_1 - u_2) = f - f = 0$$

Also whenever $|x| \leqslant 1$, $v(x, 0) = u_1(x, 0) - u_2(x, 0) = 0$. Similarly $\partial_t v(x, 0) = 0$ when $|x| \leqslant 1$. Thus by the previous subpart you can guarantee $v(x, t) = 0$ on the region $0 \leqslant ct \leqslant 1 - |x|$. Of course, $v = 0$ is the same as saying $u_1 = u_2$, so the two solutions u_1 and u_2 are certainly equal on the region $0 \leqslant ct \leqslant 1 - |x|$. □

[10] 3. Let u satisfy the transport equation $\partial_t u + c\partial_x u = 0$, with initial data $u(x, 0) = f(x)$. Show that $\int_{-\infty}^{\infty} |u(x, t)|\, dx$ is constant in time. [As you know very well, the absolute value function is *not* differentiable. So please don't provide a proof that differentiates the absolute value function. You can assume that $\int_{-\infty}^{\infty} |f| < \infty$.]

Solution. We know that $u(x, t) = f(x - ct)$. Thus $\int_{-\infty}^{\infty} |u(x, t)|\, dx = \int_{-\infty}^{\infty} |f(x - ct)|\, dx = \int_{-\infty}^{\infty} |f(y)| dy$, by making the change of variable $y = x - ct$. Of course $\int_{-\infty}^{\infty} |f(y)|\, dy$ is constant in time, concluding the proof. □

Remark. Again, you should have known this before you started the algebra. Recall this is the *transport* equation. Which, as we saw in class, just transports the initial data. So after time t, the profile of u, is just the initial profile, shifted by ct. If you just shift the profile of a function, you certainly don't change the integral!

[9] 4. (a) Let a and b be two functions such that $a(x, t) \geqslant 0$ for all x, t. Suppose u satisfies $u_t + bu_x - au_{xx} < 0$ on the rectangle $[0, L] \times [0, T]$. Show that u satisfies the strong maximum principle: Namely show that u does not attain it's maximum on the interior, or top of the rectangle $[0, L] \times [0, T]$. [Hint: The argument is mostly the same as the proof we had in class. You need to make sure nothing goes wrong with our proof when the diffusion coefficient is not constant (but still positive), and you have to figure out what to do with the bu_x term.]

Solution. Let's follow the proof of the maximum principle. Say u attained it's maximum at some interior point (x_0, t_0). Then all first order partials of u at this point are 0, and the second order partials are negative or 0. Thus at (x_0, t_0).

$$u_t + bu_x - au_{xx} = 0 + b \cdot 0 - a(\text{something negative}) \geqslant 0$$

This contradicts $u_t + bu_x - au_{xx} < 0$.
Similarly if the maximum is attained at the top, say at (x_0, T), then the only change is that we now only know that $u_t \geqslant 0$. Thus we have

$$u_t + bu_x - au_{xx} \geqslant 0 + b \cdot 0 - a(\text{something negative}) \geqslant 0$$

which is still a contradiction. □

[1] (b) What part of your proof in the previous subpart *fails* if you do not assume $a(x, t) \geqslant 0$.

Solution. We used the fact that $-a$ times something negative was positive (or 0). This is only true if $a \geqslant 0$. (It makes no difference however if a is a constant, or a function of x and t. Just so long as it is always positive (or 0).) □

[10] 5. Suppose u satisfies the heat equation $\partial_t u - \frac{1}{2}\partial_{xx} u = 0$ on $\mathbb{R}$, with initial data $u(x, 0) = f(x)$ and vanishes at infinity (i.e. $\lim_{x \to \pm\infty} u(x, t) = 0$ for any t). Assume f is such that $\int_{-\infty}^{\infty} |f| < \infty$. Show that $\lim_{t \to \infty} u(x, t) = 0$. [Hint: You can assume that u is given by the explicit formula we had in class. Use this formula to show that for some suitable constant c, we have $|u(x, t)| < \frac{c}{\sqrt{t}}$ for any $t > 0$. As an (unrelated) remark about the physical significance, the conclusion of this problem is 'natural': If you supply a *finite* amount of heat to an *infinitely* long rod, eventually it should all dissipate and the temperature should become uniformly 0.]

Solution. You should first read the fine print above, and convince yourself that this is indeed to be expected, and natural. Now for the proof: We know

$$|u(x,t)| = \left| \int f(y)G(x-y,t)\,dy \right| \leqslant \int |f(y)||G(x-y,t)|\,dy$$

Now we know $G(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$. Thus certainly $|G(x-y,t)| \leqslant \frac{1}{\sqrt{2\pi t}}$. Hence

$$|u(x,t)| \leqslant \frac{1}{\sqrt{2\pi t}} \int |f|$$

which immediately implies $\lim t \to \infty u(x,t) = 0$ for any x. In fact this shows that $u(x,t)$ converges to 0 *uniformly*. □

Remark. This should remind you of the proof of the Maximum principle I did in class: There we had

$$u(x,t) = \int f(y)G(x-y,t)\,dy \leqslant \max(f) \int G$$

The proof of this problem essentially uses

$$u(x,t) = \int f(y)G(x-y,t)\,dy \leqslant \max(G) \int f.$$

Of course, f need not be positive so the previous inequality is not technically correct. But if we add absolute values (as we did in our 'rigorous' proof), we get what we need.

I should remark that the rate of convergence here is of some interest: The above shows that $u(x,t)$ converges to 0 roughly like $\frac{1}{\sqrt{t}}$ as $t \to \infty$. You might wonder what happens in a bounded domain: Suppose you solved the heat equation on the interval $[0,L]$, with boundary boundary conditions $u(0,t) = u(L,t) = 0$ for all t. What happens as $t \to \infty$?

Well $u(x,t)$ still converges to 0 uniformly (a fact we will prove later in class). However in this case it converges to 0 *exponentially fast*. The rate of convergence can be found explicitly, and is roughly $e^{-t\sqrt{L}}$.

If you wanted to get even more general, and considered the heat equation on a region $D \subseteq \mathbb{R}^n$, with Dirichlet boundary conditions, then exactly the same conclusion above is true. u converges to 0 exponentially fast. Except you replace $\sqrt{L}$ with the smallest eigenvalue of $-\triangle$! (This is sometimes called the 'Fourier Law'.)

Remark. You might also wonder where we used the fact that u vanishes at infinity. We assumed here that $u(x,t) = \int f(y)G(x-y,t)\,dy$. This is only valid provided f was 'nice', and the boundary conditions we impose are that u vanishes at infinity. Of course, in my hint I told you to assume directly that $u(x,t) = \int f(y)G(x-y,t)\,dy$, so this was not an issue.