MATH 131 FINAL

Spring 2001

Name: ______________________________

Student ID: ___________________________

Signature: ____________________________

Instructions: Print your name and student ID number and write your signature to indicate that you accept the honor code. Read each question carefully, and show all your work. There are 12 questions. Point values are given in parentheses. Some Fourier series expansions are listed on the last page for reference. You have 3 hours to complete the exam.

Question	Score	Maximum
1		8
2		8
3		6
4		10
5		9
6		8
7		9
8		10
9		10
10		8
11		8
12		6
Total		100

1. Consider the first order equation $u_x + xu_y = 0$.

   (a) (5 points) Find the general solution.

   (b) (3 points) Which auxiliary condition leads to a unique solution, $u(x,0) = h(x)$ or $u(0,y) = h(y)$? Explain.

2. (8 points) Find the solution of the wave equation on the whole line with initial data

$$\phi(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

and $\psi(x) = 0$. Express your answer explicitly for each t as a piecewise defined function of x. Hint: Consider the intervals $x < -ct$, $-ct < x < ct$ and $x > ct$.

3. (6 points) Consider the two-dimensional wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$ and assume u is a solution of the form $u(x, y, t) = U(x, y)T(t)$. Find an ODE that T must satisfy and a PDE that U must satisfy. You do not need to solve these equations.

4. Consider the following eigenvalue problem.

$$\begin{aligned} -X'' &= \lambda X \\ X(0) &= -X(L) \\ X'(0) &= -X'(L) \end{aligned}$$

(a) (6 points) Find all positive eigenvalues and the corresponding eigenfunctions.

(b) (4 points) Show that the boundary conditions are symmetric.

5. Consider the sequence of functions $f_n(x) = e^{-nx}$ on the interval $[0, 1]$.

(a) (3 points) Show that f_n converges to the zero function in L^2 on $[0, 1]$.

(b) (3 points) To what function does f_n converge pointwise on $[0, 1]$?

(c) (3 points) Does f_n converge uniformly?

6. (8 points) The eigenfunctions for the eigenvalue problem

$$\begin{aligned} -X'' &= \lambda X \\ X(0) &= 0 \\ X'(L) &= 0 \end{aligned}$$

are

$$X_n(x) = \sin((n + \tfrac{1}{2})\pi x/L)$$

(You do not need to show this.) Find the Fourier series expansion of $f(x) = 1$ on $(0, L)$ in terms of these functions.

7. Consider the Fourier sine series for x^2 on the interval $(0, 2)$. (You do not need to compute it.)

   (a) (3 points) To what number does this series converge at $x = 2$?

   (b) (3 points) To what number does this series converge at $x = 3$?

   (c) (3 points) To what number does this series converge at $x = 5$?

8. For each function below, answer A,B,C or D. No explanation is necessary.

   (A) The full Fourier series for f converges uniformly to f on $[-\pi, \pi]$.

   (B) The full Fourier series for f converges pointwise to f on $(-\pi, \pi)$, but not uniformly.

   (C) The full Fourier series for f converges pointwise to $\frac{1}{2}[f(x+) + f(x-)]$ on $(-\pi, \pi)$, but not uniformly or pointwise to f.

   (D) None of the above hold.

   (a) (2 points) $f(x) = \tan x$

   (b) (2 points) $f(x) = e^x$

   (c) (2 points) $f(x) = e^{-|x|}$

   (d) (2 points) $f(x) = \begin{cases} \pi & x < 0 \\ x & x \geq 0 \end{cases}$

   (e) (2 points) $f(x) = \sin(x/2)$

9. Consider the following problem.

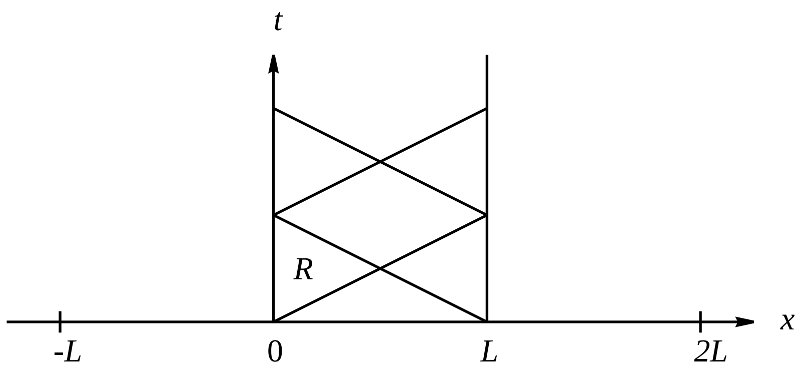
$$\begin{aligned} &u_{tt} = c^2 u_{xx} && 0 < x < L, 0 < t < \infty \\ &u(0,t) = u(L,t) = 0 && 0 < t < \infty \\ &u(x,0) = x(L-x), u_t(x,0) = 0 && 0 < x < L \end{aligned}$$

(a) (5 points) The Fourier sine series of $x(L-x)$ on $(0, L)$ is

$$\sum_{n \text{ odd}} \frac{8L}{n^3\pi^3} \sin(n\pi x/L)$$

Use this to write a series solution of the above problem. Give explicit formulas for the coefficients.

(b) (5 points) Use the method of reflection to write an explicit formula (not a series) for the solution in the region R shown below.



10. Consider the following Neumann problem for the diffusion equation.

$$
\begin{aligned}
&u_t = k u_{xx} && 0 < x < 1, 0 < t < \infty \\
&u_x(0,t) = u_x(1,t) = 0 && 0 < t < \infty \\
&u(x,0) = x && 0 < x < 1
\end{aligned}
$$

(a) (5 points) Express the solution as a series. Give explicit formulas for the coefficients.

(b) (3 points) What is the limit of the solution as t goes to infinity?

11. (8 points) Find the solution of the following inhomogeneous equation. Express the solution as a series. Give explicit formulas for the coefficients.

$$
\begin{array}{ll}
u_t = u_{xx} + e^{-t} & 0 < x < 1, 0 < t < \infty \\
u(0,t) = u(1,t) = 0 & 0 < t < \infty \\
u(x,0) = 0 & 0 < x < 1
\end{array}
$$

12. Consider the problem

$$\begin{aligned} &u_t = ku_{xx} + f(x,t) && 0 < x < L, 0 < t < \infty \\ &u(0,t) = g(t),\ u_x(L,t) = h(t) && 0 < t < \infty \\ &u(x,0) = \phi(x) && 0 < x < L \end{aligned}$$

with Dirichlet data on the left and Neumann data on the right.

(a) (4 points) Find a function $w(x,t)$ such that $v(x,t) = u(x,t) - w(x,t)$ satisfies the homogeneous boundary conditions $v(0,t) = v_x(L,t) = 0$. Hint: Try a function which is linear in x.

(b) (2 points) Write down the PDE and initial conditions satisfied by v.

Some Fourier sine and cosine series on $(0, L)$

$$1 = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(n\pi x/L)$$

$$x = \frac{L}{2} - \sum_{n \text{ odd}} \frac{4L}{n^2\pi^2} \cos(n\pi x/L)$$

$$x = \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \sin(n\pi x/L)$$

$$x^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2(-1)^n}{\pi^2 n^2} \cos(n\pi x/L)$$

$$x^2 = \sum_{n=1}^{\infty} \left(-\frac{2L^2}{n\pi}(-1)^n + \frac{4L^2}{n^3\pi^3}[(-1)^n - 1] \right) \sin(n\pi x/L)$$