Math 131 Final Examination

March 20, 2006

Name____

Signature______________________________________

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Total	

Directions:

1. This is closed book/notes exam (only the formula sheet allowed).
2. Your signature above indicates that you accept the University Honor Code.
3. Show your work on the exam sheet; you may use the back side of a page if you run out of space.
4. This test is 3 hours long and has 7 problems worth a total of 180 pts.
5. Good luck!

Problem 1. Consider the inhomogeneous wave equation $u_{tt} - u_{xx} = 2$ on the whole line.

(a) (10 pts) Find the solution of this equation with initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

(b) (5 pts) Find all stationary (time-independent) solutions.

(c) (5 pts) Suppose $u_0(x)$ is any stationary solution. Show that if the initial conditions are given by $u(x,0) = u_0(x) + \phi(x)$ and $u_t(x,0) = \psi(x)$ where ϕ and ψ are zero outside an interval, then the solution of the initial value problem converges as t goes to infinity to a stationary solution. Under what conditions is the limit solution equal to u_0?

Problem 2. Consider the diffusion equation $u_t = u_{xx}$ on the whole line with initial condition $u(x,0) = x^2$.

(a) (5 pts) Write the solution to this problem as an integral involving the diffusion kernel.

(b) (5 pts) Determine all solutions of the equation $u_t = u_{xx}$ which are of the form $u(x,t) = a_0(t) + a_1(t)x + a_2(t)x^2$.

(c) (10 pts) Find an explicit solution of the initial value problem of the form given in part (b), and use it along with the solution of part (a) to evaluate $\int_{-\infty}^{\infty} y^2 e^{-y^2}\, dy$. (Assume uniqueness for the initial value problem.)

Problem 3. Four short response questions:

(a) (8 pts) Consider the diffusion equation $u_t = ku_{xx}$ on $[0, l]$ with initial condition $u(x, 0) = \phi(x)$ and with a symmetric boundary condition. Suppose there is a negative eigenvalue λ_0 for the associated eigenvalue problem. Explain the significance for solutions of the diffusion equation?

(b) (7 pts) Determine the coefficients A_1 and A_2 so that the function $A_1 \sin(x) + A_2 \sin(2x)$ is as close as possible to the function $\phi = 1$ in the L^2 norm on $[0, \pi]$.

(c) (7 pts) In part (b), determine the L^2 distance between your answer and the function $\phi = 1$.

(d) (8 pts) Suppose a (twice continuously differentiable) function $f(x)$ on $[0, \pi]$ satisfies the Neumann boundary conditions $f'(0) = c_1$ and $f'(\pi) = c_2$. Express the n-th Fourier cosine coefficient of f'' in terms of the n-th coefficient of f and c_1 and c_2.

Problem 4. Use the convergence theorems (if they apply) to determine whether the indicated Fourier series converge pointwise, uniformly, or in L^2 (answer all three for each series). If the series converges pointwise, determine its limit for points in the indicated interval. Explain your answers, but do NOT compute the series. If a theorem is not applicable, explain why.

(a) (10 pts) The Fourier sine series for the function e^x on $[0, \pi]$.

(b) (10 pts) The full Fourier series for the function x^2 on $[-\pi, \pi]$.

(c) (5 pts) The Fourier cosine series for the function $x^{-1/3}$ on $[0, \pi]$.

(d) (5 pts)The Fourier sine series for the function $x^{-1/2}$ on $[0, \pi]$.

Problem 5. Consider the eigenvalue problem $-X'' = \lambda X$ on $[0, \pi]$ with boundary conditions $X'(0) = 0$, $X(\pi) = 0$.

(a) (10 pts) Determine all eigenvalues and corresponding eigenfunctions.

(b) (10 pts) Show that the eigenfuctions form an orthogonal set (you may quote theorems), and determine a formula for the Fourier coefficients of a function f in terms of this orthogonal set of functions.

(c) (5 pts) Show that if f is a piecewise continuous (hence square integrable) function, then the Fourier coefficients converge to zero.

Problem 6. Consider the wave equation $u_{tt} - u_{xx} = f(x,t)$ for $0 \leq x \leq \pi$ with initial conditions $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$ and homogeneous boundary conditions $u_x(0,t) = 0$, $u(\pi,t) = 0$.

(a) (10 pts) Use the results of Problem 5 to represent the solution in series form for the case $f = 0$ (homogeneous), $\phi(x) = 1$, and $\psi(x) = 0$.

(b) (10 pts) For a general function $f(x,t)$, consider the initial conditions $u(x,0) = 0$, $u_t(x,0) = 0$ and derive a sequence of ordinary differential equations for the coefficients of the solution in terms of the (time dependent) Fourier coefficients of $f(x,t)$. Find the initial conditions for these ODE's, but do not solve them.

(c) (10 pts) Assume that $f(x,t)$ is independent of t. Describe how to obtain a stationary (independent of t) solution $u_0(x)$ of the equation, and show how to reduce the inhomogeneous problem to a homogeneous one. Be sure to state the initial conditions for the new problem in terms of the old initial conditions, ϕ and ψ.

Problem 7. Let $f(x)$ be a continuously differentiable function on $[0, l]$ with $f(0) = f(l) = 0$. Let A_n for $n = 1, 2, 3, \ldots$ be the Fourier sine coefficients of f, and let α_n for $n = 0, 1, 2, \ldots$ be the Fourier cosine coefficients of f'.

(a) (5 pts) Show that $\alpha_0 = 0$, and find the relationship between A_n and α_n for $n = 1, 2, 3, \ldots$.

(b) (5 pts) Use Parseval's formula to express $\int_0^l (f(x))^2\, dx$ and $\int_0^l (f'(x))^2\, dx$ as infinite series.

(c) (10 pts) Show that the inequality

$$\int_0^l (f(x))^2\, dx \leq \left(\frac{l}{\pi}\right)^2 \int_0^l (f'(x))^2\, dx$$

holds for all such functions f.

(d) (5 pts) Determine all functions for which equality holds in the above inequality.