Assignment 6: Assigned Thu 02/15. Due Thu 02/22

1. Show that $(V^\perp)^\perp = V$.
2. Let V be a subspace of $\mathbb{R}^n$. In class we showed that any $\vec{x}$ can be expressed as $\vec{x}^\perp + \vec{x}^\parallel$, where $\vec{x}^\perp \in V^\perp$ and $\vec{x}^\parallel \in V$. Show that this expression is unique: Namely if $\vec{x} = \vec{y}_1 + \vec{z}_1 = \vec{y}_2 + \vec{z}_2$, where $\vec{y}_1, \vec{y}_2 \in V$ and $\vec{z}_1, \vec{z}_2 \in V^\perp$ then $\vec{y}_1 = \vec{y}_2$ and $\vec{z}_1 = \vec{z}_2$.
3. (a) Let V be a subspace of $\mathbb{R}^n$, and $\{v_1, \dots, v_k\}$ an orthonormal basis. Let $\vec{x} \in \mathbb{R}^n$, and $\vec{x}^\parallel$ denote the orthogonal projection of $\vec{x}$ onto V, and $\vec{x}^\perp = \vec{x} - \vec{x}^\parallel$. Show that $\vec{x}^\parallel = \sum_{i=1}^k (\vec{x} \cdot \vec{v}_i)\vec{v}_i$. [I stated but did not prove this in class. Prove this similar to the one dimensional example I gave you in class: Let $\vec{y} = \sum_{i=1}^k (\vec{x} \cdot \vec{v}_i)\vec{v}_i$ and $\vec{z} = \vec{x} - \vec{y}$, and check the same three properties I did in class.]

   (b) If in the previous subpart $\{\vec{v}_1, \dots \vec{v}_k\}$ was an orthogonal basis, show that $\vec{x}^\parallel = \sum_{i=1}^k \frac{\vec{x}\cdot\vec{v}_i}{\|\vec{v}_i\|} \frac{\vec{v}_i}{\|\vec{v}_i\|}$.

   The crucial ingredient of the formula $\vec{x}^\parallel = \sum_{i=1}^k (\vec{x} \cdot \vec{v}_i)\vec{v}_i$ is the orthogonality of $\{\vec{v}_1, \dots, \vec{v}_k\}$. If the normality assumption is dropped (like subpart (b)), then the formula still holds, provided you scale the coefficients as in subpart (b). However if the orthogonality assumption is dropped, the formula can not be modified to work.

   (c) Give an example of a subspace V of $\mathbb{R}^n$, a basis $\{v_1, \dots, v_k\}$ of V, and a vector $\vec{x} \in \mathbb{R}^n$ such that the formula for the parallel part (in the previous subpart) is not valid.
4. Let $V \subseteq \mathbb{R}^3$ be the orthogonal complement of $\operatorname{span}\{\vec{e}_1 + \vec{e}_2 + \vec{e}_3\}$. Let $\vec{x} \in \mathbb{R}^3$. Find an explicit formula for $\vec{x}^\perp$ and $\vec{x}^\parallel$ (in terms of the coordinates of $\vec{x}$), where $\vec{x}^\perp \in V^\perp$, $\vec{x}^\parallel \in V$, and $\vec{x} = \vec{x}^\parallel + \vec{x}^\perp$.
5. **Section 5.1** 11, 14, 22, 24, 26, 36

Assignment 7: Assigned Thu 02/22. Due Thu 03/01

1. Show that $(\ker A)^\perp = \operatorname{im}(A^t)$.
2. Let $d, a_1, \dots, a_n \in \mathbb{R}$, and P be the 'hyperplane' defined by $P = \{\vec{x} \in \mathbb{R}^n \mid a_1 x_1 + \cdots + a_n x_n = d\}$. Show that the shortest distance between P and the origin is $\frac{d}{\sqrt{a_1^2 + \cdots a_n^2}}$. [Hint: First choose $n = 2$, and draw a picture. Then try and generalize your argument.]
3. (a) Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, and $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n$ be such that $\vec{x} = \sum_i \alpha_i \vec{v}_i$ and $\vec{y} = \sum_i \beta_i \vec{v}_i$. Show that $\vec{x} \cdot \vec{y} = \sum_i \alpha_i \beta_i$. [Notice that $(\vec{x})_B = \vec{\alpha}$, and $(\vec{y})_B = \vec{\beta}$. This problem says that the dot product $\vec{x} \cdot \vec{y}$, where $\vec{x}$ and $\vec{y}$ are in standard coordinates is the same as $(\vec{x})_B \cdot (\vec{y})_B$, where coordinates are in basis B.]

   (b) Give an example to show that $\vec{x} \cdot \vec{y} \neq (\vec{x})_B \cdot (\vec{y})_B$ if B is not orthonormal.
4. (a) Let A be an $m \times n$ matrix, and $\vec{b} \in \mathbb{R}^m$. Let $V = \operatorname{im} A$ (note that V is a subspace of $\mathbb{R}^m$), and suppose $\ker A = \{\vec{0}\}$. Show that $P_V(\vec{b}) = A(A^t A)^{-1} A^t \vec{b}$. [Hint: Least squares ...]

   (b) Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of some subspace $V \subseteq \mathbb{R}^m$, and A be the $m \times n$ matrix with the vectors $\vec{v}_i$'s as columns. Find a matrix M (in terms of the matrix A) such that $P_V(\vec{b}) = Mb$. [This tells you how to find the matrix of an orthogonal projection when you have a basis which is *not* orthogonal. In reality however, it is better to use Gram-Schmidt to convert your basis to an orthonormal basis, and then use the formula from class]
5. I never gave an example of Least squares in class. I partially remedy that here (see also examples in the book). Suppose we want to find a line $y = c_0 + c_1 x$ that 'best fits' the points $(0,0)$, $(1,1)$, $(1,2)$.

   (a) If indeed there was a line that passed through the three points above, show that the coefficients c_0 and c_1 satisfy $A\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \vec{b}$, where $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, and $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

   (b) With A and $\vec{b}$ as above, show that the system $A\vec{x} = \vec{b}$ has no solutions.

   (c) Let $\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$ be the least square solution to $A\vec{x} = \vec{b}$. Sketch the line $y = c_0 + c_1 x$ and the points $(0,0)$, $(1,1)$, $(1,2)$ on the same graph.
6. **Section 5.2:** 32 [Preferably find a basis first, and then use Gram-Schmidt.]
7. **Section 5.3:** 4, 32
8. **Section 5.4:** 8, 10